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Monetary and Economic Department

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# Inflation risk premia in the US and the euro area*

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November 2010

**Abstract**

We use a joint model of macroeconomic and term structure dynamics to estimate inflation risk premia in the United States and the euro area. To sharpen our estimation, we include in the information set macro data and survey data on inflation and interest rate expectations at various future horizons, as well as term structure data from both nominal and index-linked bonds. Our results show that, in both currency areas, inflation risk premia are relatively small, positive, and increasing in maturity. The cyclical dynamics of long-term inflation risk premia are mostly associated with changes in output gaps, while their high-frequency fluctuations seem to be aligned with variations in inflation. However, the cyclicality of inflation premia differs between the US and the euro area. Long term inflation premia are countercyclical in the euro area, while they are procyclical in the US.

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## 1 Introduction

As markets for inflation-linked securities have grown in recent years, prices of such securities have increasingly become an important source of information about the state of the economy for market participants as well as central banks and other public institutions. Index-linked bonds, for example, provide a means of measuring ex ante real yields at different maturities. In combination with nominal yields, observable from markets for nominal bonds, real rates also allow us to calculate a "break-even inflation rate", i.e. the rate of inflation for which the pay-off from the two types of bonds would be equal.

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In practice, the break-even inflation rate is often approximated by the simple difference between a nominal yield and a real yield of similar time to maturity. Because of their timeliness and simplicity, break-even inflation rates are seen as useful indicators of the markets' expectations of future inflation. Moreover, implied forward break-even inflation rates for distant horizons are often viewed as providing information about the credibility of the central bank's commitment to maintaining price stability.

Of course, break-even rates do not, in general, reflect only inflation expectations. They also include risk premia, notably to compensate investors for inflation risk, and possibly to compensate for differential liquidity risk in the nominal and index-linked bond markets. Such premia complicate the interpretation of break-even rates as measures of inflation expectations. In this paper we focus on modelling and estimating the first of these two components - i.e. the inflation risk premium - in order to obtain a "cleaner" measure of investors' inflation expectations embedded in bond prices. In doing so, we try to reduce the risk that liquidity factors might distort our estimates by carefully choosing when to introduce yields on index-linked bonds in the estimations. We also include survey information on expectations, which should aid us in pinning down the dynamics of key variables in the model. Moreover, in order to understand the macroeconomic determinants of inflation risk premia we employ a joint model of macroeconomic and term structure dynamics, such that prices of real and nominal bonds are determined by the macroeconomic framework and investors' risk characteristics. More specifically, building on Ang and Piazzesi (2003), we adopt the framework developed in Hördahl, Tristani and Vestin (2006), in which bonds are priced based on the dynamics of the short rate obtained from the solution of a linear forward-looking macro model and using an essentially affine stochastic discount factor (see Duffie and Kan, 1996; Dai and Singleton, 2000; Duffee, 2002).

We estimate our model on US and euro area data. This provides us with an opportunity to examine the main features of inflation risk premia for the two largest economies, including similarities and differences in determinants of such premia. Our results show that the inflation risk premium is relatively small, but positive, and increasing with the maturity, in the United States as well as in the euro area.

Due to our use of term structure, survey and macroeconomic data in our estimation, we provide new empirical evidence on risk premia for the two major currency areas. Our results are broadly line with those of some recent papers, including Christensen et al. (2010) and Chernov and Mueller (2008).[1] However, compared to these papers, as well

[1] Our findings differ to some extent from the results of other studies. For example, Buraschi and Jiltsov (2005) find, within a monetary version of a real business cycle model, that the 10-year US inflation risk premium has averaged 70 basis points during a 40-year period since 1960. Using an essentially affine term structure model with regime switching, Ang, Bekaert and Wei (2006) also find a large and highly time-varying inflation risk premium, which for the 5-year horizon (the longest considered by them) averages around 115 basis points over their 1952−2004 sample. The larger size of inflation risk premia in these

as Durham (2006) and D'Amico et al. (2008), who estimate affine models on US data, our inclusion of data on economic activity as well as inflation allows us to analyse the dynamic relationship between risk premia and the economic cycle. The same is true in comparison to Garcia and Werner (2008), who study inflation risk premia in the euro area. Our results show that, in both economic areas, fluctuations in inflation premia tend to be associated with movements in the output gap and inflation. The business cycle movements in long-term inflation risk premia largely match those of the output gap, while the more high-frequency premia fluctuations seem to be aligned with changes in the level of inflation.

There is however one striking difference in the conditional dynamics of risk premia in the two currency areas. While we find that inflation premia always respond positively to upward inflation shocks, the response to output gap shocks differ between the US and the euro area. A positive output shock results in a higher inflation premium in the US, while it lowers it in the euro area. The positive relationship for the US could reflect perceptions of a higher risk of inflation surprises on the upside as the output gap widens, while the euro area result is consistent with investors becoming more willing to take on risks - including inflation risks - during booms, while they may require larger premia during recessions.

The aforementioned results on the dynamics of risk premia in the euro area are broadly in line with the results in Hördahl and Tristani (2010), who estimate a model similar to the one in this paper on euro area data. Compared to that paper, our estimates of the market prices of risk are disciplined by the inclusion of survey data in the econometric analysis.

The rest of this paper is organized as follows. The next section describes our model, its implications for the inflation risk premium and the econometric methodology, while Section 3 discusses the data. The empirical results are presented in Section 4, where we show our parameter estimates and their implications for term premia and inflation risk premia. In this section, we also relate premia to their macroeconomic determinants and calculate premium-adjusted break-even inflation rates. Section 5 concludes the paper.

## 2 The model

We rely on a simple economic model in the new-Keynesian tradition and specified directly at the aggregate level. The model includes a forward looking Phillips curve (e.g. Galí

studies may however be due simply to the focus on a longer sample period characterised by higher levels and volatility of inflation. Moreover, these studies do not include data on index-linked bonds in the estimation

Chen et al. (2005) do include index-linked bond yields in the estimation and obtain large positive inflation risk premia, averaging around 90 basis points for the 10-year maturity. However, they rely on a 2-factor CIR model with a restrictive price of risk specification, which results in sizeable pricing errors.

and Gertler, 1999) and a consumption-Euler equations (e.g. Fuhrer, 2000). Compared to the alternative of using a microfounded model, the advantage of this approach is that it imposes milder theoretical constraints. This flexibility allows us to provide descriptive evidence on the dynamics of risk premia, conditional on a widely used law of motion for macroeconomic variables and on the assumption of rational expectations. The evidence, in turn, can be interpreted as a stylised fact that successful microfounded models should be able to match. The flexibility, however, comes at a price: in the absence of a microfounded stochastic discount factor, we are unable to explain why certain risks appear to be priced more than others from an empirical viewpoint.

The specification of the model is similar to that in Hördahl, Tristani and Vestin (2006), and we therefore describe it only very briefly here. The model includes two equations which describe the evolution of inflation, $\pi_t$ in deviation from its mean $\bar{\pi}$, and the output gap, $x_t$:

$$\pi_t = \frac{\mu_\pi}{12}\sum_{i=1}^{12} E_t\left[\pi_{t+i}\right] + (1-\mu_\pi)\sum_{i=1}^{2}\delta_{\pi,i}\pi_{t\ \ i} + \delta_x x_t + \varepsilon_t^\pi, \tag{1}$$

$$x_t = \frac{\mu_x}{12}\sum_{i=1}^{12} E_t\left[x_{t+i}\right] + (1-\mu_x)\sum_{i=1}^{2}\zeta_{x,i}x_{t\ \ i} - \zeta_r\left(r_t - E_t\left[\pi_{t+1}\right]\right) + \varepsilon_t^x, \tag{2}$$

where $r_t$ is the one-month nominal interest rate (in deviation from its mean $\bar{r}$), inflation is defined as the year-on-year change in the log-price level. The lead and lag structure reflects the fact that we will be using monthly data for the estimations.

In this setup, inflation can be due to demand shocks $\varepsilon_t^x$, which increase output above potential and create excess demand, and to cost-push shocks $\varepsilon_t^\pi$, which have a direct impact on prices. In addition, monetary policy can affect inflation by changing the real interest rate $r_t - E_t\left[\pi_{t+1}\right]$, or influencing expectations. Specifically, we assume that the central bank sets the nominal short rate according to a forward-looking rule of a type proposed by Taylor (1993):

$$r_t = (1-\rho)\left\{\beta\left(E_t\left[\pi_{t+11}\right] - \pi_{t\ }\right) + \gamma x_t\right\} + \rho r_{t\ \ 1} + \eta_t \tag{3}$$

where $\pi_{t\ }$ is the perceived inflation target and $\eta_t$ is a monetary policy shock. The inflation target is allowed to be time-varying to allow for some evolution in the behavior of monetary policy over time, or at least in the way monetary policy was perceived by markets. The coefficient $\rho$ captures interest rate smoothing, reflecting the central bank's desire to avoid producing large volatility in nominal interest rates.

Finally, we assume that the monetary policy shock is serially uncorrelated, while the

other structural shocks may be correlated.[2] All shocks are assumed normally distributed with constant variance. The unobservable inflation target, is assumed to follow an AR(1) process

$$\pi_t = (1 - \phi_{\pi^*})\,\bar{\pi} + \phi_{\pi^*}\pi_{t\ \ 1} + u_{\pi^*,t} \tag{4}$$

where $u_{\pi^*,t}$ is a normal disturbance with constant variance uncorrelated with the other structural shocks.[3]

In order to obtain the rational expectations solution to the model, we write it in state-space form and proceed to solve it numerically (see Appendix A.1). The result are two matrices $\mathbf{M}$ and $\mathbf{C}$ which give the law of motion of the predetermined variables $\mathbf{X}_{1,t} = [x_{t\ \ 1}, x_{t\ \ 2}, x_{t\ \ 3}, \pi_{t\ \ 1}, \pi_{t\ \ 2}, \pi_{t\ \ 3}, \pi_t, \eta_t, \varepsilon_t^{\pi}, \varepsilon_t^{x}, r_{t\ \ 1}]^0$, and which describe how the non-predetermined variables $\mathbf{X}_{2,t} = [E_t x_{t+11}, ..., E_t x_{t+1}, x_t, E_t \pi_{t+11}, ..., E_t \pi_{t+1}, \pi_t]^0$ depend on $\mathbf{X}_{1,t}$. Specifically, we obtain

$$\mathbf{X}_{1,t} = \mathbf{M}\mathbf{X}_{1,t\ \ 1} + \Sigma\xi_{1,t}, \tag{5}$$

$$\mathbf{X}_{2,t} = \mathbf{C}\mathbf{X}_{1,t}, \tag{6}$$

where $\xi_1$ is a vector of independent, normally distributed shocks, and we also get an expression for the equilibrium short-term interest rate in terms of the state variables: $r_t = \mathbf{\Delta}^0\mathbf{X}_{1,t}$.

Because the state variables follow a first-order Gaussian VAR and the short-term interest rate is expressed as a linear function of the state vector, we can derive bond prices once we impose the assumption of absence of arbitrage opportunities and specify a process for the pricing kernel. Appendix A.2 describes the pricing kernel we choose. One important aspect here is the choice of the market prices of risk. Following Duffee (2002), we assume that these risk prices are affine functions of the states, or, more precisely, affine functions of a particular transformation of the original states. Specifically, we define the transformed state vector $\mathbf{Z}_t \equiv [x_{t\ \ 1}, x_{t\ \ 2}, x_{t\ \ 3}, \pi_{t\ \ 1}, \pi_{t\ \ 2}, \pi_{t\ \ 3}, \pi_t, r_t, \pi_t, x_t, r_{t\ \ 1}]^0$, which is obtained as $\mathbf{Z}_t = \hat{\mathbf{D}}\mathbf{X}_{1,t}$ for a suitably defined matrix $\hat{\mathbf{D}}$. Working with $\mathbf{Z}_t$ rather than $\mathbf{X}_{1,t}$ facilitates interpretation of the results later on. Our specification for the market prices of risk therefore has the following form

$$\lambda_t = \lambda_0 + \lambda_1\mathbf{Z}_t, \tag{7}$$

[2] In principle, the monetary policy shock could also be allowed to be serially correlated. We assume instead that the interest rate smoothing component, in combination with the other persistent elements in the policy rule, will be sufficient to capture all of the persistence in the short rate.

[3] To ensure stationarity of the inflation target process, we impose an upper limit of 0.99 on the $\phi_{\pi^*}$ parameter during the estimation process. This restriction is binding.

so that the risk premium associated with each of the four shocks in $\xi_{1,t}$ can depend on the level of all the state variables. To keep the number of parameters manageable, we allow only the $4 \times 4$ sub-matrix in $\lambda_1$ corresponding to the non-lagged states $[\pi_t, r_t, \pi_t, x_t]$ in $\mathbf{Z}_t$ to be non-zero.

Given the setup described above, we can write the continuously compounded yield $y_t^n$ on a zero coupon nominal bond with maturity $n$ as

$$y_t^n = A_n + B_n^{\prime} \mathbf{Z}_t, \tag{8}$$

where the $A_n$ and $B_n^{\prime}$ matrices can be derived using recursive relations (see Appendix A.2). Stacking all yields in a vector $\mathbf{Y}_t$, we write the above equations jointly as $\mathbf{Y}_t = \mathbf{A} + \mathbf{B}\mathbf{Z}_t$ or, equivalently, $\mathbf{Y}_t = \mathbf{A} + \tilde{\mathbf{B}}\mathbf{X}_{1,t}$, where $\tilde{\mathbf{B}} \equiv \mathbf{B}\hat{\mathbf{D}}$. Similarly, for real bonds ${y_t}^n$ we obtain

$${y_t}^n = A_n + B_n^{\prime}\, \mathbf{Z}_t, \tag{9}$$

and $\mathbf{Y}_t = \mathbf{A} + \tilde{\mathbf{B}}\ \mathbf{X}_{1,t}$, with $\tilde{\mathbf{B}} \equiv \mathbf{B}\ \hat{\mathbf{D}}$.

Given the solutions for real and nominal bonds, we can derive the inflation risk premium as the difference between historical and risk-adjusted expectations of future inflation rates. In so doing, we follow closely Hördahl and Tristani (2010) – see also Appendix A.4.

## 2.1 Estimation

We will evaluate the model likelihood using the Kalman filter. We first define a vector $\mathbf{W}_t$ containing the observable contemporaneous variables,

$$\mathbf{W}_t \equiv \begin{bmatrix} \mathbf{Y}_t \\ \mathbf{Y}_t \\ \mathbf{X}_{2,t}^o \\ \mathbf{U}_t \end{bmatrix},$$

where $\mathbf{Y}_t$ and $\mathbf{Y}_t$ denote vectors of nominal and real zero-coupon yields, $\mathbf{X}_{2,t}^o = [x_t, \pi_t]^{\prime}$ contains the macro variables, and where $\mathbf{U}_t$ denotes survey expectations that are included in the estimation (see below). The dimension of $\mathbf{W}_t$ is denoted $n_y$. Next, we partition the vector of predetermined variables into observable ($\mathbf{X}_{1,t}^o$) and unobservable variables ($\mathbf{X}_{1,t}^u$) according to

$$\begin{aligned} \mathbf{X}_{1,t}^o &= [x_{t\ \ 1}, x_{t\ \ 2}, x_{t\ \ 3}, \pi_{t\ \ 1}, \pi_{t\ \ 2}, \pi_{t\ \ 3}, r_{t\ \ 1}]^{\prime}, \\ \mathbf{X}_{1,t}^u &= [\pi_t, \eta_t, \varepsilon_t^{\pi}, \varepsilon_t^{x}]^{\prime}. \end{aligned}$$

To define the observation equation, we note that the interest rate and inflation expectations reflected in the survey data can be written as suitable linear functions of the states:

$$\mathbf{U}_t = \mathbf{G}\mathbf{X}_{1,t}.$$

For example, for an inflation forecast $j$ periods ahead, we know, based on the model solution, that $E_t\left[\pi_{t+j}\right] = \mathbf{C}_\pi E_t\left[\mathbf{X}_{1,t+j}\right] = \mathbf{C}_\pi \mathbf{M}^j \mathbf{X}_{1,t}$, where $\mathbf{C}_\pi$ picks the row in $\mathbf{C}$ corresponding to the inflation equation. Thus, if $\mathbf{U}_t$ contained survey data reflecting this expectation, the corresponding row in $\mathbf{G}$ would be defined as $\mathbf{C}_\pi \mathbf{M}^j$. Similar expressions would be derived for the interest rate survey forecasts included in the estimations (the specific survey data used is discussed in the next section).[4]

Given the model solution, the nominal and real bond pricing equations, and the expressions for the survey forecasts, we can proceed to define the observation equation as

$$\begin{aligned}
\mathbf{W}_t &= \begin{bmatrix} \mathbf{A} \\ \mathbf{A} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{B}} \\ \tilde{\mathbf{B}} \\ \mathbf{C} \\ \mathbf{G} \end{bmatrix} \mathbf{X}_{1,t} \\
&= \begin{bmatrix} \mathbf{A} \\ \mathbf{A} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{B}}^o \\ \tilde{\mathbf{B}}^o \\ \mathbf{C}^o \\ \mathbf{G}^o \end{bmatrix} \mathbf{X}^o_{1,t} + \begin{bmatrix} \tilde{\mathbf{B}}^u \\ \tilde{\mathbf{B}}^u \\ \mathbf{C}^u \\ \mathbf{G}^u \end{bmatrix} \mathbf{X}^u_{1,t} \\
&\equiv \mathbf{K} + \mathbf{L}^0 \mathbf{X}^o_{1,t} + \mathbf{H}^0 \mathbf{X}^u_{1,t},
\end{aligned}$$

and the measurement equation as

$$\mathbf{X}^u_{1,t} = \mathbf{F}\mathbf{X}^u_{1,t\ \ 1} + \mathbf{v}^u_{1,t},$$

where $\mathbf{F}$ selects the sub-matrix of $\mathbf{M}$ corresponding to $\mathbf{X}^u_1$.

Next, the unobservable variables are estimated using the Kalman filter. In doing so, we first introduce a vector $\mathbf{w}_t$ of serially uncorrelated measurement errors corresponding to the observable variables $\mathbf{W}_t$. Letting $\mathbf{R}$ denote the variance-covariance matrix of the measurement errors and $\mathbf{Q}$ the variances of the unobservable state variables $\mathbf{X}^u_{1,t}$, we have

$$\begin{aligned}
\mathbf{W}_t &= \mathbf{K} + \mathbf{L}^0 \mathbf{X}^o_{1,t} + \mathbf{H}^0 \mathbf{X}^u_{1,t} + \mathbf{w}_t, \qquad \mathrm{E}\left[\mathbf{w}_t \mathbf{w}_t^0\right] = \mathbf{R} \\
\mathbf{X}^u_{1,t} &= \mathbf{F}\mathbf{X}^u_{1,t\ \ 1} + \mathbf{v}^u_{1,t}, \qquad \mathrm{E}\left[\mathbf{v}^u_{1,t} \mathbf{v}^{u0}_{1,t}\right] = \mathbf{Q}.
\end{aligned}$$

[4] When defining $\mathbf{G}$, we of course take into account that some of the survey forecasts refer to average expectations over e.g. a quarter or over several years.

While we assume that all observable variables are subject to measurement error, we limit the number of parameters to estimate by assuming that all yield measurement errors have identical variance, and that all errors are mutually uncorrelated:

$$\mathbf{R} = \begin{bmatrix} \sigma^2_{m,y} & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma^2_{m,y} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2_{m,x} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma^2_{m,\pi} & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots \end{bmatrix}.$$

Note also that

$$\mathbf{Q} = \begin{bmatrix} \sigma^2_{\pi^*} & 0 & 0 & 0 \\ 0 & \sigma^2_{\eta} & 0 & 0 \\ 0 & 0 & \sigma^2_{\pi} & 0 \\ 0 & 0 & 0 & \sigma^2_{x} \end{bmatrix}.$$

We start the filter from the unconditional mean and the unconditional MSE matrix (see Hamilton, 1994). The Kalman filter produces forecasts of the states and the associated MSE, which feed into the likelihood function. Given the large number of parameters involved in the estimation, we do not directly maximise the likelihood, but we introduce priors and proceed by relying on Bayesian estimation methods. Specifically, we maximise the posterior density function obtained by combining the log-likelihood function with the prior density for the model parameters. An advantage of such an approach is that it allows us to exploit prior information on structural economic relationships available from previous studies. Moreover, the inclusion of prior distributions brings an added advantage in that it tends to make the optimisation of the highly nonlinear estimation problem more stable.

Our assumptions regarding the prior distribution of the macro parameters, which are broadly in line with those implied by typical calibrations of the new-Keynesian model, are listed in Tables 1 and 2. For the parameters that determine the market prices of risk (the elements in $\lambda_0$ and $\lambda_1$) we assume normal priors centered at zero with very large standard errors, reflecting our lack of prior information regarding these parameters. We proceed to find the mode of the posterior density using the simulated annealing algorithm (see Goffe, Ferrier and Rogers, 1994) and simulate the posterior by drawing from a distribution centered at the mode using the Metropolis-Hastings sampling algorithm (see An and Schorfheide, 2007).[5]

[5] The estimations were performed using modified versions of Frank Schorfheide's Gauss code for Bayesian

# 3 Data

We estimate the model using monthly data on nominal and real zero-coupon Treasury yields, inflation, a measure of the output gap, and survey expectations of the short-term interest rate and inflation. The model is applied to US and euro area data. To avoid obvious structural break issues associated with the introduction of the single currency, we limit our euro area sample to the period January 1999 - April 2008. For the US, we include more historical data and start our sample in January 1990.

We treat the yields on index-linked bonds are reflecting risk-free real yields, i.e. we assume that the inflation risk borne by investors because of the indexation lag (the fact that there exists a lag between the publication of the inflation index and the indexation of the bond) is negligible. Evans (1998) estimates the indexation-lag premium for UK index-linked bonds, and finds that it is likely to be quite small, around 1.5 basis points. Since the indexation lag in the UK is 8 months, while the lag in the US and the euro area is only 2.5 - 3 months, it seems likely that any indexation-lag premium for these two markets would be even smaller than Evan's estimate.

In addition to the aforementioned premium, the indexation lag can induce deviations in index-linked yields away from the true underlying real yield due to inflation seasonality and to "carry" effects. Inflation seasonality matters because index-linked bonds are linked to the seasonally unadjusted price level, which means that bond prices will be affected due to the indexation lag, unless the seasonal effect at a given date is identical to that corresponding to the indexation lag (which is in general not the case; see e.g. Ejsing et al., 2007). The carry effect refers to the fact that often index-linked yields contain some amount of realized inflation, due to predictable changes in inflation during the indexation lag period (see D'Amico et al., 2008, for a discussion of the carry effect). While these lag effects can be sizeable for short-dated bonds, they tend to be quite small for longer-term bonds. In this paper, we abstract from such effects, as they are likely to be of second-order importance for our purposes. By excluding short-term real yields (below 3 years) in the estimations, we reduce the risk that index-lag effects might influence our results to any significant extent.

## 3.1 US data

The US real and nominal term structure data consists of zero-coupon yields based on the Nelson-Siegel-Svensson (NSS) method, which are available from the Federal Reserve Board.[6] For the nominal bonds, seven maturities ranging from one month to 10 years are

estimation of DSGE models. His original code is available at http://www.econ.upenn.edu/~schorf/

[6] This data is described in detail in Gürkaynat et al. (2007, 2010).

used in the estimation, while for the real bonds we include four maturities from three to 10 years (Figures 1a and 2a).

While nominal yield data is available from the beginning of the sample, real zero-coupon yields can be obtained only from 1999. Moreover, due to well-known liquidity problems in the TIPS market during the first few years after its creation, we include real yields in the estimation only as of 2003. D'Amico et al. (2008) provide a lengthy discussion on the illiquidity of the TIPS market in the early years and argue that it resulted in severe distortions in TIPS yields. In order to reduce the risk that our estimates are biased by such distortions, we therefore exclude the first few years of real yield data. From a practical point of view, this amounts to treating them as unobservable variables prior to 2003, and to include them in the measurement equation only thereafter.[7]

Our inflation data is *y-o-y* CPI (seasonally adjusted) log-differences, observed at a monthly frequency and scaled by 12 to get an approximate monthly measure, while the output gap is computed as the log-difference of real GDP and the Congressional Budget Office's estimate of potential real GDP, which is a quarterly series. Since we estimate the model using a monthly frequency, and since the output gap is a state variable, we need a monthly series for the gap. This is obtained by fitting an ARMA(1,1) model to the quarterly gap series, forecasting the gap one quarter ahead, and computing one- and two-month ahead values by means of linear interpolation. This exercise is conducted in "real time", in the sense that the model is reestimated at each quarter using data only up to that quarter.

Following Kim and Orphanides (2005) we also use data on survey forecasts for inflation and the three-month interest rate in the estimations, obtained from the Philadelphia Fed's quarterly Survey of Professional Forecasters.[8] As argued by Kim and Orphanides (2005), survey data is likely to contain useful information for pinning down the dynamics of the state variables that determine the bond yields, which, due to the high persistence of interest rates, is a challenging task. For the US, we include six survey series: the expected 3-month interest rate two quarters ahead, four quarters ahead and during the coming 10 years, and the expected CPI inflation for the same horizons. These survey forecasts are available at a quarterly frequency, with the exception of the 10-year forecast of the 3-month interest rate which is reported only once per year. The surveys therefore enter the measurement equation only in those months when they are released.

[7] D'Amico et al. (2008) experiment with a number of different options: in one version of their estimations they exclude TIPS altogether, in another they include TIPS as of 1999, and in a third version TIPS yields enter the estimation only as of 2005.

[8] D'Amico et al. (2008) discuss at length various survey forecasts available for the United States. They conclude that inflation surveys based on the forecasts of business economists, such as the SPF, are preferable to consumer surveys.

### 3.2 Euro area data

The data setup for the euro area is similar to that for the US. We use nominal and real zero-coupon yields for the same maturities as in the US case (Figure 1b and 2b). The nominal yields are based on the NSS method applied to German data, as reported by the Bundesbank. For large parts of the maturity spectrum, the German nominal bond market is seen as the benchmark for the euro area. For the real yields, we estimate the zero-coupon rates using NSS, based on prices of AAA-rated euro area government bonds linked to the euro area HICP, issued by Germany and France (obtained from Bloomberg). We focus on AAA-rated bonds and exclude HICP-linked bonds issued by Italy and Greece (with AA- and A rating, respectively) to avoid mixing bonds with different credit ratings. Moreover, the French segment of the market is the largest in the euro area, which suggests that liquidity conditions in this market are likely to be relatively good.

The first HICP-linked government bond was issued by the French Treasury in November 2001, and the issuance of additional bonds by France, and later Germany, was very gradual. For this reason, we were able to estimate a euro area real zero coupon curve only as of January 2004, which is the date as of which we include real yields in the estimation of our model.[9] The fact that we do not include the first years in the estimation is likely to reduce potential effects on our estimates arising from initial illiquid conditions in the index-linked market, similar to the US case. Moreover, prior to the introduction of HICP-linked bonds, a market for French bonds linked to the French CPI had been growing since 1998, which may have had a positive impact on the overall liquidity situation for the euro area index-linked bond market.

As in the US case, our measure of inflation is monthly *y-o-y* HICP log-differences. Because there is no official estimate of euro area potential GDP, we follow Clarida, Galí and Gertler (1998) and measure the output gap as deviations of real GDP from a quadratic trend. This is calculated in "real time", i.e. estimated at each point in time using only information available up to that point, and monthly values are obtained using the same forecasting/interpolation method as in the case of the US output gap.

The euro area survey data we include in the estimation consists of forecasts for inflation obtained from the ECB's quarterly Survey of Professional Forecasters, and three-month interest rate forecasts available on a monthly basis from Consensus Economics. The inflation forecasts refer to expectations of HICP inflation one, two, and five years ahead. The survey data for the short-term interest rate correspond to forecasts three and 12 months ahead.

[9] Due to data limitations at the beginning of the sample, we included in the calculation of real zeros one A+ rated bond issued by the Italian Treasury during the first 10 months.

# 4 Empirical results

Tables 1 and 2 report parameter estimates and associated posterior distributions for the US as well as the euro area, respectively. The results show that our model seems empirically plausible, with estimated macro parameters that are broadly within the range of estimates which can be found in the literature (see e.g. Rudebusch, 2002; Smets and Wouters, 2003).

In both sets of estimates, the policy rule is characterized by a high degree of interest rate smoothing ($\rho$), however more so for the euro area than in the case of the US. This might reflect the shorter sample used in the estimation of the euro area model, during which the short rate remained relatively stable. The responses to inflation deviations from the objective and the output gap ($\beta$ and $\gamma$ respectively) are estimated to be similar in the two economies, and also in line with typical values reported in the literature. The degree of forward-lookingness of the output gap equation and the inflation equation is somewhat higher for the euro area than for the US. As for the estimated standard deviations of fundamental shocks, these are generally higher for the US than for the euro area. This is likely to be due to the relatively low macroeconomic volatility during the euro sample, compared to the longer sample available for the US.

As already mentioned, our assumed perceived policy rule allows for a time-varying inflation target. This is an unobservable variable that needs to be filtered out from available observable data. Figures 4a and 4b display the estimates obtained for the US and the euro area, respectively. From an intuitive viewpoint, these estimates seem reasonable: in both cases the filtered target moves slowly and with little variability compared to realised inflation. The US target estimate shows more movement than the euro area one, fluctuating slowly within an interval between approximately 2.5% and 3.5%. In comparison, the euro area target is nearly constant just below the 2% level. This difference may be partly due to the availability of an official numerical definition of price stability in the euro area, and partly to the greater variability of actual inflation in the longer US sample.

## 4.1 Term premia and inflation risk premia

Given a set of parameters and a specific realisation of the state variable vector, our model implies a nominal term premium for any maturity, as well as a decomposition of the nominal premium into a real premium and an inflation premium.[10] The dynamics of our estimated nominal and inflation premia are displayed in Figure 5, with a focus on the 10-year maturity. The US 10-year nominal term premium has displayed a near-secular decline during the sample period, dropping from a level close to 3% to almost zero in

[10] Here, we disregard the component due to Jensen's inequality, which is in the order of only a few basis points.

recent years – a feature that has also been found by D'Amico et al. (2008), among others. Our results indicate that, to a large extent, this decline in the nominal premium has been due to a falling real premium. At the same time, the US inflation premium also seems to have fallen somewhat in recent years, following a sharp drop in the first couple of years of the new millennium. This drop coincided with a sharp fall in US inflation and growing concerns about deflationary pressures in the wake of sharp declines in equity prices and an economic downturn. Overall, the magnitude of our estimated inflation premia are comparable two recent empirical evidence reported by Chernov and Mueller (2008), and Christensen et al. (2010).

The estimates of long-term nominal premia in the euro area show that these have fallen in line with US term premia. Again, much of this has been attributable to declining real premia, while the inflation premium has remained relatively more stable around a small positive mean. This is in line with the results in Hördahl and Tristani (2010).

### 4.2 Premium-adjusted break-even inflation rates

Given our estimates of the inflation risk premium, we can strip out this component from observable break-even inflation rates to obtain premium-adjusted break-even inflation rates, which provide a model-consistent measure of inflation expectations over the life of the bonds. Figure 6 reports raw and premium-adjusted 10-year break-even inflation rates in the US and the euro area for the period during which we have reliable estimates of zero-coupon real rates (see the data section above). Reflecting the relatively small magnitude of our estimated premia, the raw and adjusted break-even rates tend to be close to one another. With euro area inflation premia estimated to be somewhat larger than in the US on average, the euro area adjusted break-even rate is consequently also lower relative to the raw rate. In fact, while the raw euro area break-even rate has been fluctuating consistently above a level of 2% since 2004, the premium-adjusted measure has been close to and mostly below 2%, suggesting long-term euro area inflation expectations more in line with the ECB's price stability objective than would have been the case had one taken the unadjusted break-even rate to represent expected inflation.

Figure 6 also displays the estimated model-implied average expected inflation over the next 10 years for each point in time during the sample periods. This is the expected inflation produced by the macro dynamics of the model, which would fully coincide with the premium-adjusted break-even rate discussed above if all yield measurement errors were always zero. While this is not the case, the difference is always very small, in the order of a few basis points, indicating that our model does well in terms of capturing the dynamics of both nominal and real yields.

Finally, Figure 6 reports measures of long-horizon inflation expectations from available

survey forecasts: 10-year US inflation expectations from the Fed's Survey of Professional Forecasters (SPF) and 5-year euro area inflation expectations from the ECB's SPF. Clearly, inclusion of inflation survey data in the estimation has been useful in getting the model to capture the broad movements of investors' inflation expectations, as reported by these survey measures. Moreover, in the case of the euro area, where the premium-adjusted break-even rate has differed more from its raw counterpart than in the US, the adjusted break-even rate is much closer to the survey forecasts than the unadjusted rate. With respect to the US, the survey expectations displayed in Figure 6a provide some justification for the very small US inflation risk premia estimates that we obtain: since 2003, the raw US break-even rate has been relatively well aligned with the survey measure, suggesting that the inflation premium needs to be small to produce an adjusted break-even rate close to the survey expectations. While small, the fluctuations in the estimated premium that have taken place have generally resulted in a premium-adjusted break-even rate that is closer to the survey measure than the unadjusted rate.

### 4.3 The inflation risk premium and the macroeconomy

One key advantage of our modelling strategy is to allow us to relate movements in inflation risk premia to macroeconomic developments in the US and the euro area. Our results suggest that, both in the US and in the euro area, changes in inflation premia are mostly associated with changes in two (observable) macroeconomic variables: the output gap and inflation. As displayed in Figures 7 and 8, the broad movements in the 10-year inflation risk premium largely match those of the output gap, while the more high-frequency fluctuations in the premium seem to be aligned with changes in the level of inflation.

More specifically, in the case of the US, inflation risk premia tend to rise when the output gap is increasing, and vice versa (Fig. 7a), possibly reflecting perceptions of a higher risk of inflation surprises on the upside as the output gap widens. Apart from these dynamics at the cyclical frequency, there is also a positive correlation between month-to-month inflation premium changes and realised inflation (Fig. 7b). This same "high-frequency" pattern is present in the euro area (Fig. 8b), but the cyclical covariation between the euro area inflation premium and the output gap appears to be mostly negative instead of positive (Fig. 8a). With the inflation premium accounting for a sizeable portion of the overall term premium, this result seems in line with the the widely documented counter-cyclicality of term premia (see e.g. Stambaugh, 1988, Hördahl, Tristani and Vestin, 2006).[11] A possible explanation for this could be that investors become more willing to take on risks -

[11] Given this evidence, the pro-cyclical features of the estimated US inflation premium may seem puzzling. However, our estimates also show that the real premium component in US long-term bond yields is highly counter-cyclical, which may have been the driving force behind the often reported result that term premia tend to be counter-cyclical.

including inflation risks - during booms, while they require larger inflation premia during recessions.

A second advantage of our modelling strategy is to allow us to compute impulse response functions of yields and associated premia to the underlying macro shocks. Figures 9-12 show the impulse responses of a number of variables to the output gap and inflation, which are the two shocks that are most important for the inflation risk premium, for the US and the euro area. The variables selected for the responses are the break-even inflation rate, the expected inflation, and the inflation risk premium. In each figure, the three left-hand panels show responses of these variables for the two-year horizon, while the right-hand panels correspond to the 10-year horizon.

The figures confirm the pattern observed in Figures 7 and 8, in the sense that responses of all variables to output gap shocks are much more persistent than those to inflation shocks. For example, the half-life of the response of the 10-year US break-even rate to an output gap shock is around 30 months, while it is about 12 months for inflation shocks. The figures also confirm that the response of the inflation premium to inflation shocks is uniformly positive, while the premium response to the output gap varies between the US and the euro area.

Looking at the results in more detail, Figure 9 shows that a one standard deviation upward shock to the output gap (about 0.4%) in the US pushes the 10-year break-even rate up by around 15 basis points on impact. About two thirds of this effect is due to a rising inflation premium, while one third corresponds to an increase in expected inflation as a result of the expansionary shock. At the 2-year horizon, the effect on the break-even rate is even larger, at around 26 basis points on impact, but now the bulk of this response is due to rising inflation expectations (16 basis points), whereas the inflation premium accounts for the remaining 10 basis points. Hence, a shock to the output gap seems to result in a parallel shift in the inflation premium, while inflation expectations react much more strongly for short horizons than long, reflecting the short- to medium-term persistence of output gap shocks. In the euro area, a positive shock to the output gap also raises expected inflation - and more so at the 2-year horizon than the 10-year horizon - but the inflation premium response is uniformly negative (Fig. 11). Moreover, as the expected inflation response declines with the horizon, the break-even response goes from being positive at the 2-year horizon to being slightly negative at 10 years.

With respect to the responses to an inflation shock, the results are similar in both economies. A one standard deviation upward shock to inflation (about 0.15%) raises the 10-year break-even rate by around 4 basis points on impact. This is the result of both higher inflation expectations and higher inflation premia, although the latter effect dominates in the case of the euro area. For the shorter 2-year horizon, the responses are

similar but several times magnified, in line with the short duration of inflation shocks. The 2-year break-even rate jumps by 16-24 basis points, of which 12-14 basis points is due to increasing inflation premia and the rest to higher expected inflation over the next two years.

# 5 Conclusions

Break-even inflation rates are often used as timely measures of market expectations of future inflation, and are therefore viewed as useful indicators for central banks, among others. However, some care should be exercised when interpreting break-even inflation rates in terms of inflation expectations, because they include risk premia, most notably to compensate investors for inflation risk. In this paper we model and estimate the inflation risk premium in order to obtain a "cleaner" measure of investors' true inflation expectations embedded in bond prices. In addition, we investigate the macroeconomic determinants of inflation risk premia, in order to better understand their dynamics.

We estimate our model on US and euro area data. This provides us with an opportunity to examine the main features of inflation risk premia for the two largest economies, including similarities and differences in the determinants of such premia. Our results show that the inflation risk premium is relatively small, positive, and increasing with the maturity, in the United States as well as in the euro area. Our estimated inflation premia vary over time as a result of changes to the state variables in the model. Specifically, in both economies the output gap and inflation are the main drivers of inflation premia. The broad movements in long-term inflation risk premia largely match those of the output gap, while more high-frequency premia fluctuations seem to be aligned with changes in the level of inflation. While we find that inflation premia always respond positively to upward inflation shocks, the response to output gap shocks differ between the US and the euro area. A positive output shock results in a higher inflation premium in the US, while it lowers it in the euro area. The positive relationship for the US could reflect perceptions of a higher risk of inflation surprises on the upside as the output gap widens, while the euro area result is consistent with investors becoming more willing to take on risks - including inflation risks - during booms, while they may require larger premia during recessions.

# A Appendix

## A.1 Solving the model

In order to solve the model we write it in the general form

$$\begin{bmatrix} \mathbf{X}_{1,t+1} \\ E_t\mathbf{X}_{2,t+1} \end{bmatrix} = \mathbf{H} \begin{bmatrix} \mathbf{X}_{1,t} \\ \mathbf{X}_{2,t} \end{bmatrix} + \mathbf{K}r_t + \begin{bmatrix} \Sigma\xi_{1,t+1} \\ \mathbf{0} \end{bmatrix}, \tag{10}$$

where $\mathbf{X}_{1,t} = [x_{t\ \ 1}, x_{t\ \ 2}, x_{t\ \ 3}, \pi_{t\ \ 1}, \pi_{t\ \ 2}, \pi_{t\ \ 3}, \pi_t, \eta_t, \varepsilon_t^{\pi}, \varepsilon_t^{x}, r_{t\ \ 1}]^{\emptyset}$ is the vector of predetermined variables, $\mathbf{X}_{2,t} = [E_t x_{t+11}, ..., E_t x_{t+1}, x_t, E_t\pi_{t+11}, ..., E_t\pi_{t+1}, \pi_t]^{\emptyset}$ includes the variables which are not predetermined, $r_t$ is the policy instrument and $\xi_1$ is a vector of independent, normally distributed shocks. The short-term rate can be written in the feedback form

$$r_t = -\mathbf{F} \begin{bmatrix} \mathbf{X}_{1,t} \\ \mathbf{X}_{2,t} \end{bmatrix}. \tag{11}$$

The solution of the model can be obtained numerically following standard methods. We choose the methodology described in Söderlind (1999), which is based on the Schur decomposition. The result are two matrices $\mathbf{M}$ and $\mathbf{C}$ such that $\mathbf{X}_{1,t} = \mathbf{M}\mathbf{X}_{1,t\ \ 1} + \Sigma\xi_{1,t}$ and $\mathbf{X}_{2,t} = \mathbf{C}\mathbf{X}_{1,t}$.[12] Consequently, the equilibrium short-term interest rate will be equal to $r_t = \mathbf{\Delta}^{\emptyset}\mathbf{X}_{1,t}$, where $\mathbf{\Delta}^{\emptyset} \equiv -(\mathbf{F}_1 + \mathbf{F}_2\mathbf{C})$ and $\mathbf{F}_1$ and $\mathbf{F}_2$ are partitions of $\mathbf{F}$ conformable with $\mathbf{X}_{1,t}$ and $\mathbf{X}_{2,t}$.

## A.2 Pricing real and nominal bonds

To build the term structure of interest rates, we first note that the solution of the macro model is the same as that in standard affine term structure models. Specifically, the short-term interest rate is expressed as a linear function of the state vector ($\mathbf{X}_1$), which in turn follows a first-order Gaussian VAR.[13] To derive the term structure, we therefore only need to impose the assumption of absence of arbitrage opportunities, which guarantees the existence of a risk-neutral measure, and to specify a process for the stochastic discount factor, or pricing kernel.

The (nominal) pricing kernel $m_{t+1}$ is defined as $m_{t+1} = \exp(-r_t)\,\psi_{t+1}/\psi_t$, where $\psi_{t+1}$ is the Radon-Nikodym derivative assumed to follow the log-normal process $\psi_{t+1} = \psi_t \exp\left(-\frac{1}{2}\lambda_t^{\emptyset}\lambda_t - \lambda_t^{\emptyset}\xi_{1,t+1}\right)$, and where $\lambda_t$ denotes the market prices of risk. As described in Section 2, we assume that these risk prices are affine functions of a transformed state vector $\mathbf{Z}_t \equiv [x_{t\ \ 1}, x_{t\ \ 2}, x_{t\ \ 3}, \pi_{t\ \ 1}, \pi_{t\ \ 2}, \pi_{t\ \ 3}, \pi_t, r_t, \pi_t, x_t, r_{t\ \ 1}]^{\emptyset}$, defined as $\mathbf{Z}_t = \hat{\mathbf{D}}\mathbf{X}_{1,t}$ for a suitably defined matrix $\hat{\mathbf{D}}$. Given this transformation, the solution equation for the short-term interest rate can be rewritten as a function of $\mathbf{Z}_t$,

$$r_t = \overline{\mathbf{\Delta}}^{\emptyset}\mathbf{Z}_t. \tag{12}$$

[12] The presence of non-predetermined variables in the model implies that there may be multiple solutions for some parameter values. We constrain the system to be determinate in the iterative process of maximizing the likelihood function.

[13] Note, however, that in our case both the short-rate equation and the law of motion of vector $\mathbf{X}_1$ are obtained endogenously, as functions of the parameters of the macroeconomic model. This contrasts with the standard affine setup based on unobservable variables, where both the short rate equation and the law of motion of the state variables are postulated exogenously.

From the macro model solution, we also know that

$$\begin{aligned}\pi_{t+1} &= \mathbf{C}_\pi \mathbf{M}\mathbf{X}_{1,t} + \mathbf{C}_\pi \Sigma \xi_{1,t+1} \\ &= \mathbf{C}_\pi \mathbf{M}\hat{\mathbf{D}}^{\ 1}\mathbf{Z}_t + \mathbf{C}_\pi \Sigma \xi_{1,t+1},\end{aligned}$$

where $\mathbf{C}_\pi$ is the relevant row of $\mathbf{C}$.

Now assume that the real pricing kernel is $m_{t+1}$, so that the following fundamental asset pricing relation holds

$$E_t\left[m_{t+1}\left(1+R_{t+1}\right)\right] = 1,$$

where $R_{t+1}$ denotes the real return on some asset.

If we now want to price an $n$-period nominal bond, $p_t^n$, we get

$$\frac{p_t^n}{q_t} = E_t\left[m_{t+1}\frac{p_{t+1}^{n\ 1}}{q_{t+1}}\right],$$

where $q_t$ is the price level in the economy. In terms of inflation rates, $\pi_{t+1} \equiv \ln q_{t+1} - \ln q_t$, we obtain

$$p_t^n = E_t\left[m_{t+1}\frac{p_{t+1}^{n\ 1}}{\exp\left(\pi_{t+1}\right)}\right].$$

Notice that this is equivalent to postulating a nominal pricing kernel $m_{t+1} \equiv m_{t+1}/\exp\left(\pi_{t+1}\right)$, such that

$$p_t^n = E_t\left[m_{t+1}p_{t+1}^{n\ 1}\right].$$

Given our assumption on the nominal pricing kernel and on the market prices of risk, we can postulate that nominal bond prices will be exponential-affine functions of the state variables, to obtain

$$p_t^n = \exp\left(\bar{A}_n + \bar{B}_n^0\mathbf{Z}_t\right), \tag{13}$$

where $\bar{A}_n$ and $\bar{B}_n^0$ are recursive parameters that depend on the maturity $n$ in the following way:

$$\bar{A}_{n+1} = \bar{A}_n - \bar{B}_n^0\hat{\mathbf{D}}\Sigma\lambda_0 + \tfrac{1}{2}\bar{B}_n^0\hat{\mathbf{D}}\Sigma\Sigma^0\hat{\mathbf{D}}^0\bar{B}_n, \tag{14}$$

$$\bar{B}_{n+1}^0 = \bar{B}_n^0\hat{\mathbf{D}}\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right) - \overline{\mathbf{\Delta}}^0. \tag{15}$$

Nominal bond yields are then given by

$$\begin{aligned}y_t^n &= -\frac{\ln\left(p_t^n\right)}{n} \\ &= -\frac{\bar{A}_n}{n} - \frac{\bar{B}_n^0}{n}\mathbf{Z}_t \\ &\equiv A_n + B_n^0\mathbf{Z}_t.\end{aligned} \tag{16}$$

The definition of the pricing kernel implies

$$r_t = -\ln m_{t+1} - \frac{1}{2}\lambda_t^0\lambda_t - \lambda_t^0\xi_{1,t+1},$$

which translates into a real pricing kernel

$$m_{t+1} = \exp\left(-r_t + \pi_{t+1}\right) \frac{\psi_{t+1}}{\psi_t},$$

or

$$m_{t+1} = \exp\left(-\overline{\boldsymbol{\Delta}}^{\prime}\mathbf{Z}_t + \mathbf{C}_\pi \mathbf{M}\mathbf{X}_{1,t} + \mathbf{C}_\pi \Sigma \xi_{1,t+1} - \frac{1}{2}\lambda_t^{\prime}\lambda_t - \lambda_t^{\prime}\xi_{1,t+1}\right).$$

We postulate again that real bond prices will be exponential-affine functions of the state variables,

$$p_t^n = \exp\left(\bar{A}_n + \bar{B}_n^{\prime}\mathbf{Z}_t\right),$$

where $\bar{A}_n$ and $\bar{B}_n$ are parameters that depend on the maturity $n$, and which can be identified using

$$\begin{aligned} p_t^{n+1} &= E_t\left[m_{t+1}p_{t+1}^n\right] \\ &= \exp\left(\bar{A}_n - \overline{\boldsymbol{\Delta}}^{\prime}\mathbf{Z}_t + \mathbf{C}_\pi \mathbf{M}\mathbf{X}_{1,t} + \bar{B}_n^{\prime}\hat{\mathbf{D}}\mathbf{M}\mathbf{X}_{1,t} - \frac{1}{2}\lambda_t^{\prime}\lambda_t\right) \\ &\times E_t\left[\exp\left(\left(\mathbf{C}_\pi\Sigma - \lambda_t^{\prime} + \bar{B}_n^{\prime}\hat{\mathbf{D}}\Sigma\right)\xi_{1,t+1}\right)\right], \end{aligned}$$

where we used

$$\mathbf{Z}_{t+1} = \hat{\mathbf{D}}\mathbf{M}\mathbf{X}_{1,t} + \hat{\mathbf{D}}\Sigma\xi_{1,t+1}.$$

Noting that

$$E_t\left[\exp\left(\left(\mathbf{C}_\pi\Sigma + \bar{B}_n^{\prime}\hat{\mathbf{D}}\Sigma - \lambda_t^{\prime}\right)\xi_{1,t+1}\right)\right] = \exp\left(\tfrac{1}{2}\left(\left(\mathbf{C}_\pi + \bar{B}_n^{\prime}\hat{\mathbf{D}}\right)\Sigma - \lambda_t^{\prime}\right)\left(\left(\mathbf{C}_\pi + \bar{B}_n^{\prime}\hat{\mathbf{D}}\right)\Sigma - \lambda_t^{\prime}\right)^{\prime}\right),$$

and rearranging terms, we obtain

$$\begin{aligned} p_t^{n+1} &= \exp\left(\bar{A}_n + \tfrac{1}{2}\left(\mathbf{C}_\pi + \bar{B}_n^{\prime}\hat{\mathbf{D}}\right)\Sigma\Sigma^{\prime}\left(\mathbf{C}_\pi + \bar{B}_n^{\prime}\hat{\mathbf{D}}\right)^{\prime} - \left(\mathbf{C}_\pi + \bar{B}_n^{\prime}\hat{\mathbf{D}}\right)\Sigma\lambda_0\right. \\ &\left. + \left(\left(\mathbf{C}_\pi + \bar{B}_n^{\prime}\hat{\mathbf{D}}\right)\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right) - \overline{\boldsymbol{\Delta}}^{\prime}\right)\mathbf{Z}_t\right). \end{aligned}$$

We can therefore identify $\bar{A}_n$ and $\bar{B}_n$ recursively as

$$\begin{aligned} \bar{A}_{n+1} &= \bar{A}_n + \tfrac{1}{2}\left(\mathbf{C}_\pi + \bar{B}_n^{\prime}\hat{\mathbf{D}}\right)\Sigma\Sigma^{\prime}\left(\mathbf{C}_\pi + \bar{B}_n^{\prime}\hat{\mathbf{D}}\right)^{\prime} - \left(\mathbf{C}_\pi + \bar{B}_n^{\prime}\hat{\mathbf{D}}\right)\Sigma\lambda_0, \\ \bar{B}_{n+1}^{\prime} &= \left(\mathbf{C}_\pi + \bar{B}_n^{\prime}\hat{\mathbf{D}}\right)\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right) - \overline{\boldsymbol{\Delta}}^{\prime}. \end{aligned}$$

For a 1-month real bond, in particular, we obtain

$$\begin{aligned} p_t^1 &= E_t\left[m_{t+1}\right] \\ &= \exp\left(\left(-\overline{\boldsymbol{\Delta}}^{\prime} + \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right)\right)\mathbf{Z}_t - \mathbf{C}_\pi\Sigma\left(\lambda_0 - \tfrac{1}{2}\Sigma^{\prime}\mathbf{C}_\pi^{\prime}\right)\right), \end{aligned}$$

which can be used to start the recursion. Note that the short-term real rate is

$$r_t = \mathbf{C}_\pi\Sigma\left(\lambda_0 - \tfrac{1}{2}\Sigma^{\prime}\mathbf{C}_\pi^{\prime}\right) + \left(\overline{\boldsymbol{\Delta}}^{\prime} - \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right)\right)\mathbf{Z}_t.$$

## A.3 Short-rate spread

The effect of the inflation risk premium is to drive a wedge between riskless real yields and ex-ante real yields, namely nominal yields net of expected inflation. For the short-term rate, in particular, we can write

$$r_t = r_t + E_t\left[\pi_{t+1}\right] + prem_{\pi,t} + \tfrac{1}{2}\mathbf{C}_\pi \Sigma\Sigma^0\mathbf{C}_\pi^0,$$

where

$$\begin{aligned} r_t &= \mathbf{C}_\pi\Sigma\left(\lambda_0 - \tfrac{1}{2}\Sigma^0\mathbf{C}_\pi^0\right) + \left(\overline{\boldsymbol{\Delta}}^0 - \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right)\right)\mathbf{Z}_t \\ E_t\left[\pi_{t+1}\right] &= \mathbf{C}_\pi\mathbf{M}\hat{\mathbf{D}}^{\ 1}\mathbf{Z}_t \\ prem_{\pi,t} &= -\mathbf{C}_\pi\Sigma\left(\lambda_0 + \lambda_1\mathbf{Z}_t\right). \end{aligned}$$

Note that the discrepancy between ex-ante real and risk-free rates is not only due to inflation risk, but also includes a convexity term $\frac{1}{2}\mathbf{C}_\pi\Sigma\Sigma^0\mathbf{C}_\pi^0$. We define as inflation risk premium the component of the difference which would vanish if market prices of risk were zero.

## A.4 Derivation of inflation risk premium and break-even inflation rates

For all maturities, recall that the continuously compounded yield is, for nominal and real bonds, respectively

$$\begin{aligned} y_{t,n} &= -\frac{\bar{A}_n}{n} - \frac{\bar{B}_n^0}{n}\mathbf{Z}_t \\ y_{t,n} &= -\frac{\bar{A}_n}{n} - \frac{\bar{B}_n^{\ 0}}{n}\mathbf{Z}_t. \end{aligned}$$

The yield spread is therefore simply

$$y_{t,n} - y_{t,n} = -\frac{1}{n}\left(\bar{A}_n - \bar{A}_n\right) - \frac{1}{n}\left(\bar{B}_n^0 - \bar{B}_n^{\ 0}\right)\mathbf{Z}_t,$$

where

$$\begin{aligned} \bar{A}_{n+1} - \bar{A}_{n+1} &= \bar{A}_n - \bar{A}_n - \left(\bar{B}_n^0 - \bar{B}_n^{\ 0}\right)\hat{\mathbf{D}}\Sigma\lambda_0 + \mathbf{C}_\pi\Sigma\lambda_0 - \tfrac{1}{2}\mathbf{C}_\pi\Sigma\Sigma^0\mathbf{C}_\pi^0 \\ &\quad - \mathbf{C}_\pi\Sigma\Sigma^0\hat{\mathbf{D}}^0\bar{B}_n + \tfrac{1}{2}\left(\bar{B}_n^0\hat{\mathbf{D}}\Sigma\Sigma^0\hat{\mathbf{D}}^0\bar{B}_n - \bar{B}_n^{\ 0}\hat{\mathbf{D}}\Sigma\Sigma^0\hat{\mathbf{D}}^0\bar{B}_n\right) \\ \bar{B}_{n+1}^0 - \bar{B}_{n+1}^{\ 0} &= \left(\bar{B}_n^0 - \bar{B}_n^{\ 0}\right)\hat{\mathbf{D}}\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right) - \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right). \end{aligned}$$

Note that the nominal bond equation can be solved explicitly as

$$\begin{aligned} \bar{A}_n &= \bar{A}_1 + \sum_{i=1}^{n}{}^{1}\left(\tfrac{1}{2}\overline{\mathbf{B}}_i^0\hat{\mathbf{D}}\Sigma\Sigma^0\hat{\mathbf{D}}^0\overline{\mathbf{B}}_i - \overline{\mathbf{B}}_i^0\hat{\mathbf{D}}\Sigma\lambda_0\right), \\ \overline{B}_n^0 &= -\overline{\boldsymbol{\Delta}}^0\sum_{i=0}^{n}{}^{1}\left[\hat{\mathbf{D}}\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right)\right]^i. \end{aligned}$$

Similar, for the real bond $\bar{A}_n$ we obtain

$$\bar{A}_n = n\mathbf{C}_\pi \Sigma \left(\tfrac{1}{2}\Sigma^0 \mathbf{C}_\pi^0 - \lambda_0\right) + \sum_{i=1}^{n} {}^{1} \left(\bar{B}_i{}' \hat{\mathbf{D}} \Sigma \Sigma^0 \mathbf{C}_\pi^0 + \tfrac{1}{2} \bar{B}_i{}^0 \hat{\mathbf{D}} \Sigma \Sigma^0 \hat{\mathbf{D}}^0 \bar{B}_i - \bar{B}_i{}^0 \hat{\mathbf{D}} \Sigma \lambda_0\right)$$

$$\bar{B}_n{}^0 = \left(\mathbf{C}_\pi \left(\mathbf{M}\hat{\mathbf{D}}\ {}^{1} - \Sigma\lambda_1\right) - \overline{\boldsymbol{\Delta}}^0\right) \sum_{i=0}^{n} {}^{1} \left[\hat{\mathbf{D}} \left(\mathbf{M}\hat{\mathbf{D}}\ {}^{1} - \Sigma\lambda_1\right)\right]^i.$$

Note that the law of motion of the transformed state vector can be written as $\mathbf{Z}_{t+1} = \hat{\mathbf{D}}\mathbf{M}\hat{\mathbf{D}}\ {}^{1}\mathbf{Z}_t + \hat{\mathbf{D}}\Sigma\xi_{1,t+1}$, so that the term $\hat{\mathbf{D}}\left(\mathbf{M}\hat{\mathbf{D}}\ {}^{1} - \Sigma\lambda_1\right)$ represents the expected change in $\mathbf{Z}_t$ under Q. We can then define a new matrix $\widehat{\mathbf{M}} = \hat{\mathbf{D}}\left(\mathbf{M}\hat{\mathbf{D}}\ {}^{1} - \Sigma\lambda_1\right)$. Note also that the sum $\sum_{i=0}^{n} {}^{1} \widehat{\mathbf{M}}^i$ can be solved out as $\sum_{i=0}^{n} {}^{1} \widehat{\mathbf{M}}^i = \left(\mathbf{I} - \widehat{\mathbf{M}}\right)\ {}^{1} \left(\mathbf{I} - \widehat{\mathbf{M}}^n\right)$ for bonds of finite maturity.[14] Note that we could equivalently write $\sum_{i=0}^{n} {}^{1} \widehat{\mathbf{M}}^i = \left(\mathbf{I} - \widehat{\mathbf{M}}^n\right)\left(\mathbf{I} - \widehat{\mathbf{M}}\right)\ {}^{1}$.

For the state dependent component of bond prices, it follows that

$$\overline{B}_n^0 = -\overline{\boldsymbol{\Delta}}^0 \left(\mathbf{I} - \widehat{\mathbf{M}}\right)\ {}^{1} \left(\mathbf{I} - \widehat{\mathbf{M}}^n\right)$$
$$\bar{B}_n{}^0 = \left(\mathbf{C}_\pi \hat{\mathbf{D}}\ {}^{1}\widehat{\mathbf{M}} - \overline{\boldsymbol{\Delta}}^0\right) \left(\mathbf{I} - \widehat{\mathbf{M}}\right)\ {}^{1} \left(\mathbf{I} - \widehat{\mathbf{M}}^n\right),$$

and

$$\bar{B}_n{}^0 - \overline{B}_n^0 = \mathbf{C}_\pi \hat{\mathbf{D}}\ {}^{1}\widehat{\mathbf{M}} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)\ {}^{1} \left(\mathbf{I} - \widehat{\mathbf{M}}^n\right).$$

Note also that

$$E_t\left[\pi_{t+n}\right] = \mathbf{C}_\pi \mathbf{M}^n \hat{\mathbf{D}}\ {}^{1}\mathbf{Z}_t,$$

and that expected average inflation up to $t+n$, $\overline{\pi}_{t+n}$ is

$$\begin{aligned} E_t\overline{\pi}_{t+n} &= \frac{1}{n}\sum_{i=1}^{n} E_t\pi_{t+i} \\ &= \mathbf{C}_\pi \frac{\sum_{i=1}^{n} \mathbf{M}^i}{n} \hat{\mathbf{D}}\ {}^{1}\mathbf{Z}_t, \end{aligned}$$

or, writing this out explicitly,

$$E_t\overline{\pi}_{t+n} = \frac{1}{n}\mathbf{C}_\pi \left(\mathbf{I} - \mathbf{M}^n\right)\left(\mathbf{I} - \mathbf{M}\right)\ {}^{1}\mathbf{M}\hat{\mathbf{D}}\ {}^{1}\mathbf{Z}_t.$$

We are now ready to define the break even inflation rate as

$$\begin{aligned} y_{t,n} - y_{t,n} &= \frac{1}{n}\left(\bar{A}_n - \bar{A}_n\right) + \frac{1}{n}\left(\bar{B}_n{}^0 - \bar{B}_n^0\right)\mathbf{Z}_t \\ &= \frac{1}{n}\left(\bar{A}_n - \bar{A}_n\right) + \frac{1}{n}\mathbf{C}_\pi \hat{\mathbf{D}}\ {}^{1}\widehat{\mathbf{M}} \left(\mathbf{I} - \widehat{\mathbf{M}}\right)\ {}^{1} \left(\mathbf{I} - \widehat{\mathbf{M}}^n\right)\mathbf{Z}_t, \end{aligned}$$

where $\bar{A}_n$ and $\bar{A}_n$ are defined above.

[14] For bonds of infinite maturity, the sum will only be defined if all eigenvalues of $\widehat{\mathbf{M}}$ are inside the unit circle. This is not necessarily true, even if the eigenvalues of $\mathbf{M}$ are within the unit circle by construction.

The inflation risk premium can then be defined as

$$y_{t,n} - y_{t,n} - E_t\overline{\pi}_{t+n} = \frac{1}{n}\left(\bar{A}_n - \bar{A}_n\right) + \frac{1}{n}\left(\bar{B}_n^{\ 0} - \bar{B}_n^0\right)\mathbf{Z}_t - E_t\overline{\pi}_{t+n},$$

whose state-dependent component can be written explicitly as

$$\frac{1}{n}\left(\bar{B}_n^{\ 0} - \bar{B}_n^0\right)\mathbf{Z}_t - E_t\overline{\pi}_{t+n} = \frac{1}{n}\mathbf{C}_\pi\left[\hat{\mathbf{D}}^{\ 1}\widehat{\mathbf{M}}\left(\mathbf{I}-\widehat{\mathbf{M}}\right)^{\ 1}\left(\mathbf{I}-\widehat{\mathbf{M}}^n\right) - \mathbf{M}\left(\mathbf{I}-\mathbf{M}^n\right)\left(\mathbf{I}-\mathbf{M}\right)^{\ 1}\hat{\mathbf{D}}^{\ 1}\right]\mathbf{Z}_t.$$

Note that the time-varying component of the inflation risk premium is zero *at all maturities* when the $\lambda_1$ prices of risk are zero. To see this, note that for $\lambda_1 = 0$ we obtain $\widehat{\mathbf{M}} = \hat{\mathbf{D}}\mathbf{M}\hat{\mathbf{D}}^{\ 1}$, so that $\left(\hat{\mathbf{D}}\mathbf{M}\hat{\mathbf{D}}^{\ 1}\right)^n = \hat{\mathbf{D}}\mathbf{M}^n\hat{\mathbf{D}}^{\ 1}$, and

$$\begin{aligned}
\frac{1}{n}\left(\bar{B}_n^{\ 0} - \bar{B}_n^0\right)\mathbf{Z}_t - E_t\overline{\pi}_{t+n} &= \frac{1}{n}\mathbf{C}_\pi\mathbf{M}\left[\left(\mathbf{I}-\mathbf{M}\right)^{\ 1}\hat{\mathbf{D}}^{\ 1}\left(\mathbf{I}-\hat{\mathbf{D}}\mathbf{M}^n\hat{\mathbf{D}}^{\ 1}\right) - \left(\mathbf{I}-\mathbf{M}^n\right)\left(\mathbf{I}-\mathbf{M}\right)^{\ 1}\hat{\mathbf{D}}^{\ 1}\right]\mathbf{Z}_t \\
&= \frac{1}{n}\mathbf{C}_\pi\mathbf{M}\left[\left(\mathbf{I}-\mathbf{M}\right)^{\ 1}\left(\mathbf{I}-\mathbf{M}^n\right) - \left(\mathbf{I}-\mathbf{M}^n\right)\left(\mathbf{I}-\mathbf{M}\right)^{\ 1}\right]\hat{\mathbf{D}}^{\ 1}\mathbf{Z}_t \\
&= \frac{1}{n}\mathbf{C}_\pi\mathbf{M}\left[\sum_{i=0}^{n}{}^{1}\mathbf{M}^i - \sum_{i=0}^{n}{}^{1}\mathbf{M}^i\right]\hat{\mathbf{D}}^{\ 1}\mathbf{Z}_t \\
&= 0.
\end{aligned}$$

## A.5 Holding period returns

We define the one-period expected holding period return on an $n$-bond as

$$e_{n,t} = E_t\left[\ln p_{t+1}^{n\ \ 1} - \ln p_t^n\right].$$

Using the bond equations, we know that

$$p_{t+1}^{n\ \ 1} = \exp\left(\bar{A}_{n\ \ 1} + \bar{B}_n^{\ 0}{}_1\mathbf{Z}_{t+1}\right),$$

and

$$\begin{aligned}
e_{n,t} = &-\tfrac{1}{2}\left(\mathbf{C}_\pi + \bar{B}_n^{\ 0}{}_1\hat{\mathbf{D}}\right)\Sigma\Sigma^0\left(\mathbf{C}_\pi + \bar{B}_n^{\ 0}{}_1\hat{\mathbf{D}}\right)^0 + \left(\mathbf{C}_\pi + \bar{B}_n^{\ 0}{}_1\hat{\mathbf{D}}\right)\Sigma\lambda_0 \\
&+ \left(\bar{B}_n^{\ 0}{}_1\hat{\mathbf{D}}\Sigma\lambda_1 - \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right) + \overline{\boldsymbol{\Delta}}^0\right)\mathbf{Z}_t,
\end{aligned}$$

which in case of the 1-period bond collapses to

$$e_{1,t} = -\tfrac{1}{2}\mathbf{C}_\pi\Sigma\Sigma^0\mathbf{C}_\pi^0 + \mathbf{C}_\pi\Sigma\lambda_0 + \left(\overline{\boldsymbol{\Delta}}^0 - \mathbf{C}_\pi\left(\mathbf{M}\hat{\mathbf{D}}^{\ 1} - \Sigma\lambda_1\right)\right)\mathbf{Z}_t,$$

i.e. the short-term real rate.

The excess real holding period return is therefore

$$e_{n,t} - e_{1,t} = -\tfrac{1}{2}\bar{B}_n^{\ 0}{}_1\hat{\mathbf{D}}\Sigma\Sigma^0\hat{\mathbf{D}}^0\bar{B}_{n\ \ 1} + \bar{B}_n^{\ 0}{}_1\hat{\mathbf{D}}\Sigma\left(\lambda_0 - \Sigma^0\mathbf{C}_\pi^0\right) + \bar{B}_n^{\ 0}{}_1\hat{\mathbf{D}}\Sigma\lambda_1\mathbf{Z}_t.$$

Similarly, for the nominal term structure we obtain

$$
\begin{aligned}
e_{n,t} &= -\tfrac{1}{2}\bar{B}^{0}_{n\ \ 1}\hat{\mathbf{D}}\Sigma\Sigma^{0}\hat{\mathbf{D}}^{0}\bar{B}_{n\ \ 1} + \bar{B}^{0}_{n\ \ 1}\hat{\mathbf{D}}\Sigma\lambda_{0} + \left(\bar{B}^{0}_{n\ \ 1}\hat{\mathbf{D}}\Sigma\lambda_{1} + \overline{\boldsymbol{\Delta}}^{0}\right)\mathbf{Z}_{t} \\
e_{n,t} - e_{1,t} &= \bar{B}^{0}_{n\ \ 1}\hat{\mathbf{D}}\Sigma\left(\lambda_{0} - \tfrac{1}{2}\Sigma^{0}\hat{\mathbf{D}}^{0}\bar{B}_{n\ \ 1}\right) + \bar{B}^{0}_{n\ \ 1}\hat{\mathbf{D}}\Sigma\lambda_{1}\mathbf{Z}_{t},
\end{aligned}
$$

so that the nominal-real spread net of expected inflation is

$$
\begin{aligned}
e_{n,t} - e_{n,t} - E_{t}\left[\pi_{t+1}\right] = &-\tfrac{1}{2}\left(\bar{B}^{0}_{n\ \ 1}\hat{\mathbf{D}}\Sigma\Sigma^{0}\hat{\mathbf{D}}^{0}\bar{B}_{n\ \ 1} - \bar{B}_{n}^{\ 0\ \ 1}\hat{\mathbf{D}}\Sigma\Sigma^{0}\hat{\mathbf{D}}^{0}\bar{B}_{n\ \ 1}\right) \\
&+ \mathbf{C}_{\pi}\Sigma\Sigma^{0}\hat{\mathbf{D}}^{0}\bar{B}_{n\ \ 1} + \tfrac{1}{2}\mathbf{C}_{\pi}\Sigma\Sigma^{0}\mathbf{C}^{0}_{\pi} \\
&+ \left(\left(\bar{B}^{0}_{n\ \ 1} - \bar{B}_{n}^{\ 0\ \ 1}\right)\hat{\mathbf{D}} - \mathbf{C}_{\pi}\right)\left(\Sigma\lambda_{0} + \Sigma\lambda_{1}\mathbf{Z}_{t}\right).
\end{aligned}
$$

We can rewrite this using the solutions for $\bar{B}^{0}_{n\ \ 1}$ and $\bar{B}_{n}^{\ 0\ \ 1}$ to obtain

$$
\begin{aligned}
&e_{n,t} - e_{n,t} - E_{t}\left[\pi_{t+1}\right] \\
&= -\mathbf{C}_{\pi}\hat{\mathbf{D}}\ \ {}^{1}\widehat{\mathbf{M}}\left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{\ \ 1}\left(\mathbf{I} - \widehat{\mathbf{M}}^{n\ \ 1}\right)\hat{\mathbf{D}}\Sigma\Sigma^{0}\hat{\mathbf{D}}^{0}\left(\mathbf{I} - \left(\widehat{\mathbf{M}}^{0}\right)^{n\ \ 1}\right)\left(\mathbf{I} - \widehat{\mathbf{M}}^{0}\right)^{\ \ 1}\left(\overline{\boldsymbol{\Delta}} - \tfrac{1}{2}\widehat{\mathbf{M}}^{0}\left(\hat{\mathbf{D}}^{0}\right)^{\ \ 1}\mathbf{C}^{0}_{\pi}\right) \\
&+ \mathbf{C}_{\pi}\Sigma\Sigma^{0}\hat{\mathbf{D}}^{0}\bar{B}_{n\ \ 1} + \tfrac{1}{2}\mathbf{C}_{\pi}\Sigma\Sigma^{0}\mathbf{C}^{0}_{\pi} \\
&- \left(\mathbf{C}_{\pi}\hat{\mathbf{D}}\ \ {}^{1}\widehat{\mathbf{M}}\left(\mathbf{I} - \widehat{\mathbf{M}}\right)^{\ \ 1}\left(\mathbf{I} - \widehat{\mathbf{M}}^{n\ \ 1}\right)\hat{\mathbf{D}} + \mathbf{C}_{\pi}\right)\left(\Sigma\lambda_{0} + \Sigma\lambda_{1}\mathbf{Z}_{t}\right).
\end{aligned}
$$

### A.6 Forward premia

Real 1-period forward rates are defined as

$$
\begin{aligned}
f_{n,t} &= \ln p^{n}_{t} - \ln p^{n+1}_{t} \\
&= \mathbf{C}_{\pi}\Sigma\lambda_{0} - \tfrac{1}{2}\mathbf{C}_{\pi}\Sigma\Sigma^{0}\mathbf{C}^{0}_{\pi} - \bar{B}_{n}^{\ 0}\hat{\mathbf{D}}\Sigma\left(\Sigma^{0}\mathbf{C}^{0}_{\pi} - \lambda_{0}\right) - \tfrac{1}{2}\bar{B}_{n}^{\ 0}\hat{\mathbf{D}}\Sigma\Sigma^{0}\hat{\mathbf{D}}^{0}\bar{B}_{n} \\
&\quad + \left(\bar{B}_{n}^{\ 0}\left(\mathbf{I} - \widehat{\mathbf{M}}\right) - \mathbf{C}_{\pi}\hat{\mathbf{D}}\ \ {}^{1}\widehat{\mathbf{M}} + \overline{\boldsymbol{\Delta}}^{0}\right)\mathbf{Z}_{t}.
\end{aligned}
$$

Note that

$$
E_{t}r_{t+n} = \mathbf{C}_{\pi}\Sigma\left(\lambda_{0} - \tfrac{1}{2}\Sigma^{0}\mathbf{C}^{0}_{\pi}\right) + \left(\overline{\boldsymbol{\Delta}}^{0} - \mathbf{C}_{\pi}\hat{\mathbf{D}}\ \ {}^{1}\widehat{\mathbf{M}}\right)\hat{\mathbf{D}}\mathbf{M}^{n\ \ 1}\hat{\mathbf{D}}\ \ {}^{1}\mathbf{Z}_{t},
$$

so that the real forward premium is

$$
\begin{aligned}
f_{n,t} - E_{t}r_{t+n} = &-\bar{B}_{n}^{\ 0}\hat{\mathbf{D}}\Sigma\left(\Sigma^{0}\mathbf{C}^{0}_{\pi} - \lambda_{0}\right) - \tfrac{1}{2}\bar{B}_{n}^{\ 0}\hat{\mathbf{D}}\Sigma\Sigma^{0}\hat{\mathbf{D}}^{0}\bar{B}_{n} \\
&+ \left(\bar{B}_{n}^{\ 0} - \bar{B}_{n}^{\ 0}\widehat{\mathbf{M}} + \left(\overline{\boldsymbol{\Delta}}^{0} - \mathbf{C}_{\pi}\hat{\mathbf{D}}\ \ {}^{1}\widehat{\mathbf{M}}\right)\left(\mathbf{I} - \hat{\mathbf{D}}\mathbf{M}^{n\ \ 1}\hat{\mathbf{D}}\ \ {}^{1}\right)\right)\mathbf{Z}_{t}.
\end{aligned}
$$

The nominal-real forward spread is then given by

$$
\begin{aligned}
f_{n,t} - f_{n,t} = &\left(\bar{B}^{0}_{n} - \bar{B}_{n}^{\ 0}\right)\hat{\mathbf{D}}\Sigma\lambda_{0} - \tfrac{1}{2}\bar{B}^{0}_{n}\hat{\mathbf{D}}\Sigma\Sigma^{0}\hat{\mathbf{D}}^{0}\bar{B}_{n} + \tfrac{1}{2}\bar{B}_{n}^{\ 0}\hat{\mathbf{D}}\Sigma\Sigma^{0}\hat{\mathbf{D}}^{0}\bar{B}_{n} - \mathbf{C}_{\pi}\Sigma\lambda_{0} + \tfrac{1}{2}\mathbf{C}_{\pi}\Sigma\Sigma^{0}\mathbf{C}^{0}_{\pi} \\
&+ \bar{B}_{n}^{\ 0}\hat{\mathbf{D}}\Sigma\Sigma^{0}\mathbf{C}^{0}_{\pi} + \left(\bar{B}^{0}_{n} - \bar{B}_{n}^{\ 0} - \left(\bar{B}^{0}_{n} - \bar{B}_{n}^{\ 0}\right)\widehat{\mathbf{M}} + \mathbf{C}_{\pi}\hat{\mathbf{D}}\ \ {}^{1}\widehat{\mathbf{M}}\right)\mathbf{Z}_{t}.
\end{aligned}
$$

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Table 1: US parameter estimates
(Sample period: January 1990 − April 2008)

| | Prior distribution | | | Posterior mode | | Posterior distr. | | |
|---|---|---|---|---|---|---|---|---|
| Parameter | Type | Mean | St.err.[1] | Mode | St. error | 5% | Median | 95% |
| $\rho$ | Beta | 0.9 | 0.1 | 0.731 | 0.004 | 0.716 | 0.726 | 0.737 |
| $\beta$ | Normal | 1.5 | 0.1 | 1.747 | 0.005 | 1.673 | 1.747 | 1.786 |
| $\gamma$ | Normal | 0.3 | 0.1 | 0.393 | 0.005 | 0.385 | 0.420 | 0.466 |
| $\mu_\pi$ | Beta | 0.5 | 0.2 | 0.325 | 0.008 | 0.307 | 0.328 | 0.348 |
| $\delta_x$ | Gamma | 0.1 | 0.02 | 0.031 | 0.003 | 0.028 | 0.033 | 0.037 |
| $\mu_x$ | Beta | 0.5 | 0.2 | 0.160 | 0.006 | 0.144 | 0.167 | 0.190 |
| $\zeta_r$ | Gamma | 0.1 | 0.02 | 0.101 | 0.006 | 0.098 | 0.113 | 0.125 |
| $\phi_\pi$ | Normal | 0.5 | 0.2 | 0.725 | 0.005 | 0.688 | 0.709 | 0.734 |
| $\phi_x$ | Normal | 0.9 | 0.2 | 0.959 | 0.005 | 0.943 | 0.953 | 0.960 |
| $\bar{\pi} \times 1200$ | Beta | 2.5 | 0.3 | 2.640 | 0.068 | 2.450 | 2.563 | 2.687 |
| $\bar{r} \times 1200$ | Beta | 4 | 0.4 | 4.562 | 0.106 | 4.326 | 4.507 | 4.669 |
| $\sigma_{\pi^*} \times 10^3$ | Inv gamma | 0.05 | 4 | 0.048 | 0.001 | 0.036 | 0.041 | 0.049 |
| $\sigma_\eta \times 10^3$ | Inv gamma | 0.35 | 4 | 0.189 | 0.001 | 0.189 | 0.193 | 0.198 |
| $\sigma_\pi \times 10^3$ | Inv gamma | 0.3 | 4 | 0.119 | 0.000 | 0.118 | 0.119 | 0.119 |
| $\sigma_x \times 10^3$ | Inv gamma | 0.075 | 4 | 0.026 | 0.000 | 0.025 | 0.028 | 0.032 |
| $\lambda_0\,(\pi\ )$ | Normal | 0 | 100 | −0.121 | 0.003 | −0.152 | −0.126 | −0.103 |
| $\lambda_0\,(\eta)$ | Normal | 0 | 100 | −0.289 | 0.003 | −0.296 | −0.253 | −0.198 |
| $\lambda_0\,(\pi)$ | Normal | 0 | 100 | −0.097 | 0.003 | −0.103 | −0.082 | −0.064 |
| $\lambda_0\,(x)$ | Normal | 0 | 100 | −0.041 | 0.003 | −0.056 | −0.031 | 0.002 |

$\lambda_1 \times 10^{\ 2}$ : posterior distribution[2]

| | $\pi$ | $\eta$ | $\pi$ | $x$ |
|---|---|---|---|---|
| $\pi$ | −3.901<br>( 6.231 , 2.268) | 0.345<br>(0.186 , 0.523) | 1.156<br>(0.641 , 1.762) | −1.405<br>( 1.848 , 0.969) |
| $\eta$ | −1.997<br>( 3.198 , 0.828) | −5.692<br>( 5.992 , 5.392) | 1.391<br>(0.896 , 1.890) | 4.056<br>(3.722 , 4.496) |
| $\pi$ | 5.131<br>(4.551 , 6.022) | 1.966<br>(1.798 , 2.175) | −4.613<br>( 5.165 , 4.015) | −0.389<br>( 0.611 , 0.192) |
| $x$ | −6.309<br>( 8.341 , 4.929) | −0.524<br>( 0.732 , 0.354) | 2.922<br>(2.407 , 3.616) | −1.572<br>( 2.048 , 1.235) |

Median values; 5% and 95% percentiles in parentheses

1. For the Inverted gamma distribution, the degrees of freedom are indicated. 2. For all lambda parameters, the prior distrbution is Normal with mean zero and standard error 100.

Table 2: Euro area parameter estimates
(Sample period: January 1999 − April 2008)

| | Prior distribution | | | Posterior mode | | Posterior distr. | | |
|---|---|---|---|---|---|---|---|---|
| Parameter | Type | Mean | St.err.[1] | Mode | St. error | 5% | Median | 95% |
| $\rho$ | Beta | 0.9 | 0.1 | 0.976 | 0.002 | 0.972 | 0.977 | 0.980 |
| $\beta$ | Normal | 1.5 | 0.1 | 1.511 | 0.022 | 1.421 | 1.516 | 1.641 |
| $\gamma$ | Normal | 0.3 | 0.1 | 0.303 | 0.009 | 0.225 | 0.269 | 0.364 |
| $\mu_\pi$ | Beta | 0.5 | 0.2 | 0.536 | 0.004 | 0.513 | 0.552 | 0.588 |
| $\delta_x$ | Gamma | 0.1 | 0.02 | 0.037 | 0.002 | 0.033 | 0.043 | 0.052 |
| $\mu_x$ | Beta | 0.5 | 0.2 | 0.297 | 0.011 | 0.236 | 0.290 | 0.370 |
| $\zeta_r$ | Gamma | 0.1 | 0.02 | 0.069 | 0.010 | 0.055 | 0.070 | 0.090 |
| $\phi_\pi$ | Normal | 0.5 | 0.2 | 0.680 | 0.007 | 0.637 | 0.663 | 0.707 |
| $\phi_x$ | Normal | 0.9 | 0.2 | 0.972 | 0.002 | 0.963 | 0.969 | 0.975 |
| $\bar{\pi} \times 1200$ | Beta | 2 | 0.3 | 1.852 | 0.007 | 1.787 | 1.844 | 1.886 |
| $\bar{r} \times 1200$ | Beta | 4 | 0.4 | 4.303 | 0.153 | 4.078 | 4.295 | 4.504 |
| $\sigma_{\pi^*} \times 10^3$ | Inv gamma | 0.05 | 4 | 0.011 | 0.000 | 0.011 | 0.011 | 0.011 |
| $\sigma_\eta \times 10^3$ | Inv gamma | 0.35 | 4 | 0.129 | 0.004 | 0.121 | 0.133 | 0.159 |
| $\sigma_\pi \times 10^3$ | Inv gamma | 0.3 | 4 | 0.106 | 0.000 | 0.106 | 0.106 | 0.107 |
| $\sigma_x \times 10^3$ | Inv gamma | 0.075 | 4 | 0.026 | 0.003 | 0.022 | 0.029 | 0.038 |
| $\lambda_0(\pi\ )$ | Normal | 0 | 100 | 0.088 | 0.004 | 0.065 | 0.083 | 0.104 |
| $\lambda_0(\eta)$ | Normal | 0 | 100 | 0.779 | 0.019 | 0.681 | 0.831 | 1.168 |
| $\lambda_0(\pi)$ | Normal | 0 | 100 | −0.087 | 0.003 | −0.110 | −0.086 | −0.067 |
| $\lambda_0(x)$ | Normal | 0 | 100 | 0.162 | 0.006 | 0.154 | 0.203 | 0.242 |

$\lambda_1 \times 10^{\ 2}$ : posterior distribution[2]

| | $\pi$ | $\eta$ | $\pi$ | $x$ |
|---|---|---|---|---|
| $\pi$ | −9.299<br>( 13.319 , 6.028) | −0.354<br>( 0.564 , 0.161) | −0.618<br>( 1.066 , 0.293) | −0.776<br>( 1.091 , 0.512) |
| $\eta$ | 176.159<br>(153.939 , 196.222) | 5.813<br>(4.679 , 7.002) | −2.436<br>( 4.275 , 0.501) | −6.093<br>( 8.528 , 4.346) |
| $\pi$ | 12.239<br>(4.342 , 17.945) | −1.072<br>( 1.281 , 0.800) | −13.187<br>( 15.729 , 10.171) | 5.704<br>(4.121 , 7.034) |
| $x$ | 41.587<br>(32.647 , 50.526) | 1.455<br>(1.169 , 1.820) | 0.649<br>( 0.261 , 1.425) | −2.284<br>( 3.327 , 1.631) |

Median values; 5% and 95% percentiles in parentheses

1. For the Inverted gamma distribution, the degrees of freedom are indicated. 2. For all lambda parameters, the prior distrbution is Normal with mean zero and standard error 100.

Figure 1a: US nominal zero-coupon yields

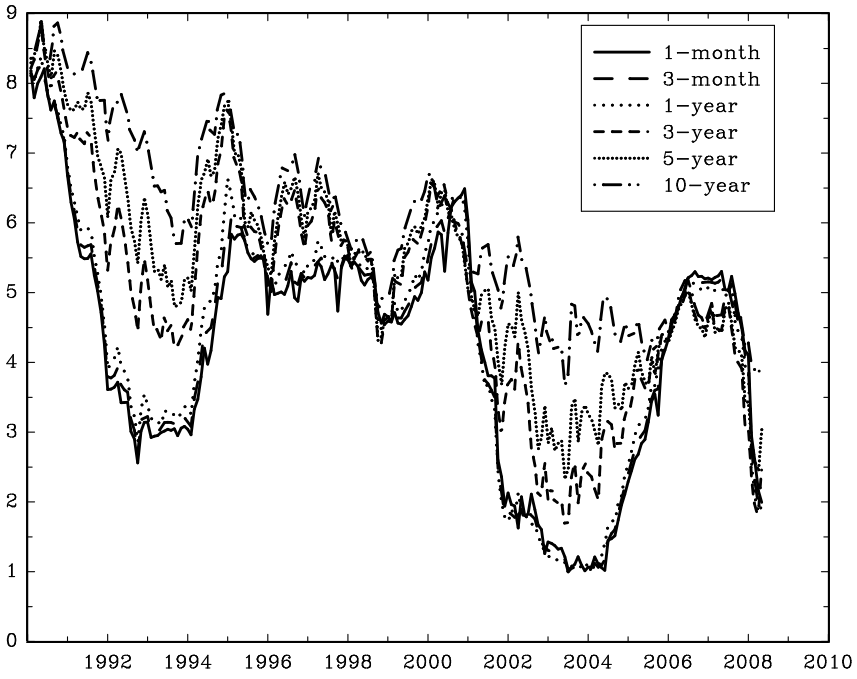



Figure 1b: Euro area nominal zero-coupon yields

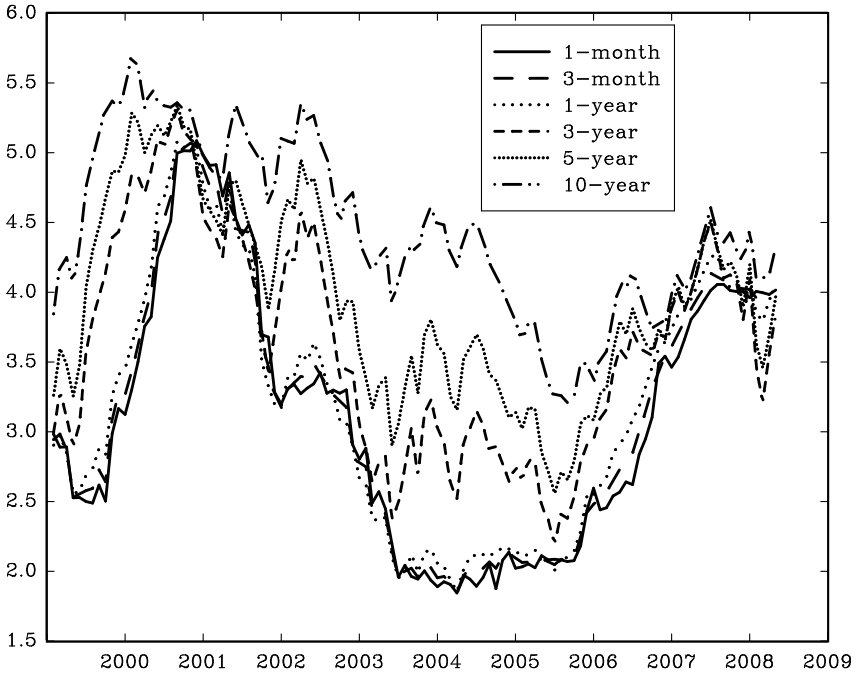


Figure 2a: US real zero-coupon yields

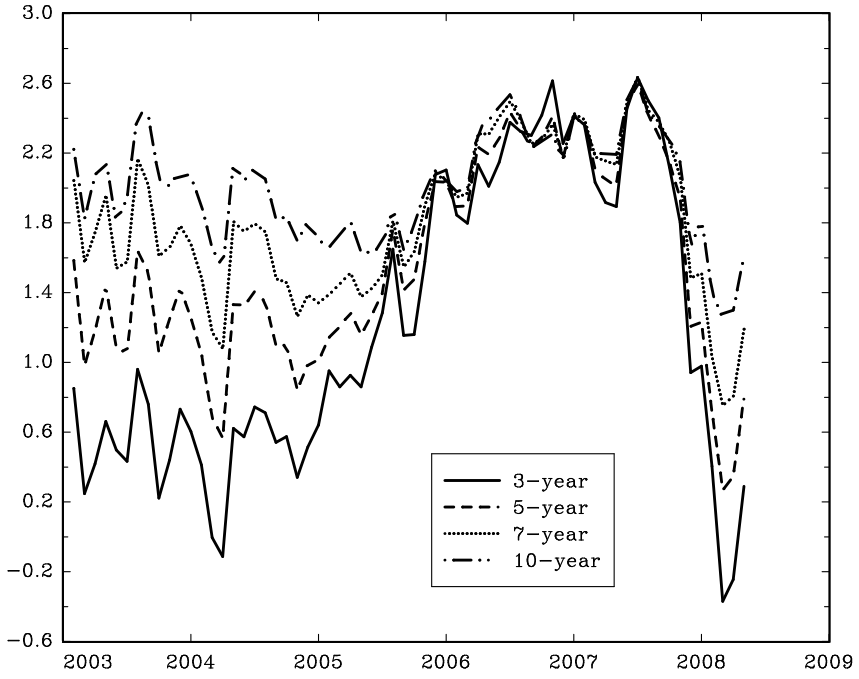



Figure 2b: Euro area real zero-coupon yields

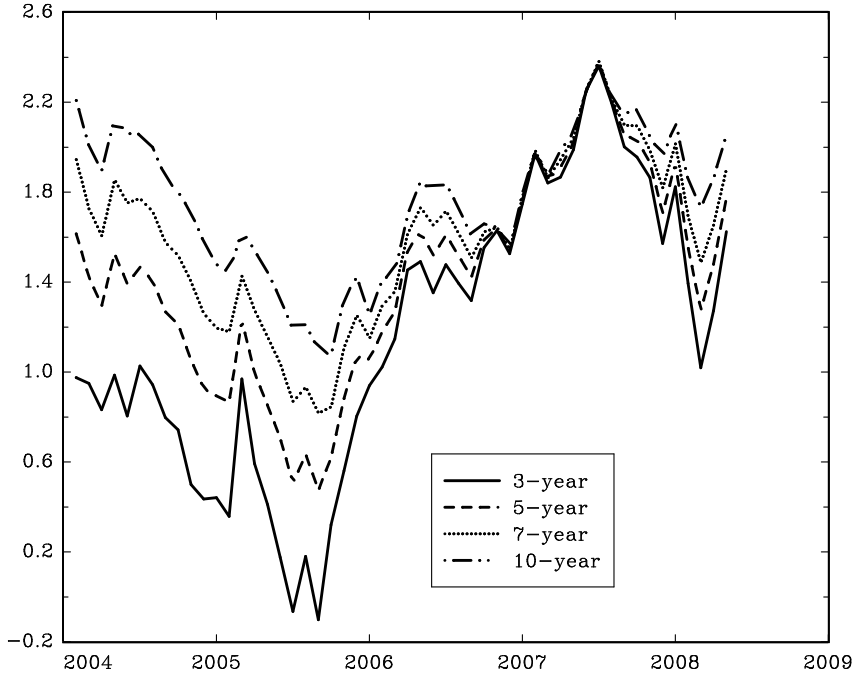
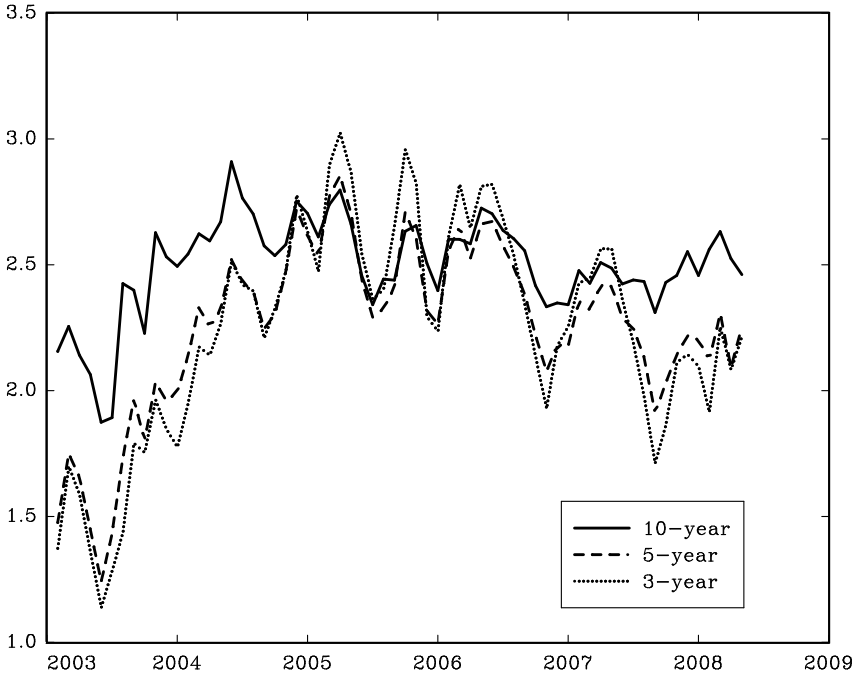
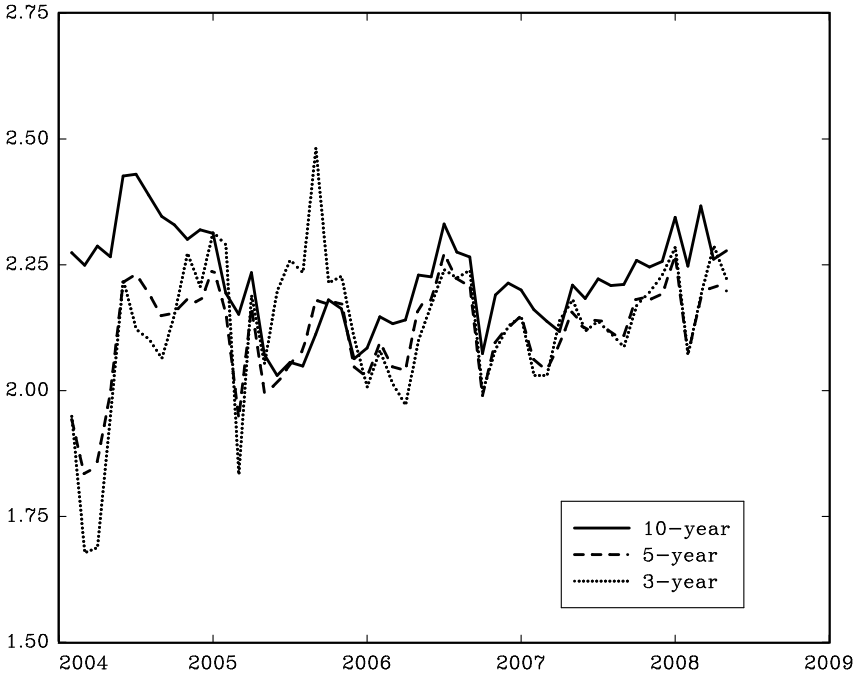


Figure 3a: US zero-coupon break-even inflation rates




Difference between model-implied nominal and real zero-coupon yields of the same maturity. Sample period: January 2003 to April 2008 (percent per year).

Figure 3b: Euro area zero-coupon break-even inflation rates




Difference between model-implied nominal and real zero-coupon yields of the same maturity. Sample period: January 2004 to April 2008 (percent per year).

Figure 4a: Inflation and estimated perceived inflation target: US

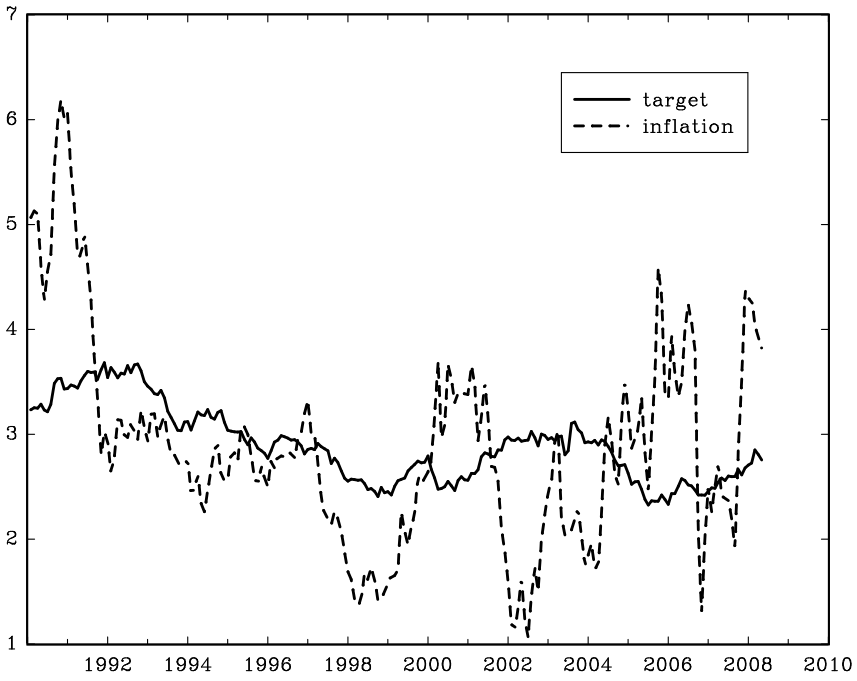



Figure 4b: Inflation and estimated perceived inflation target: euro area

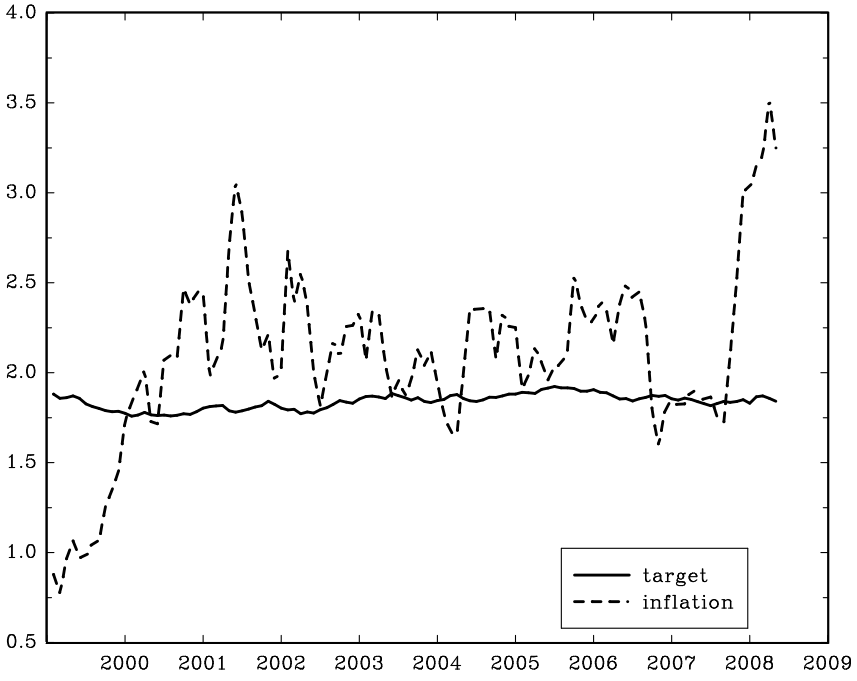


Figure 5a: Estimated 10-year US premia

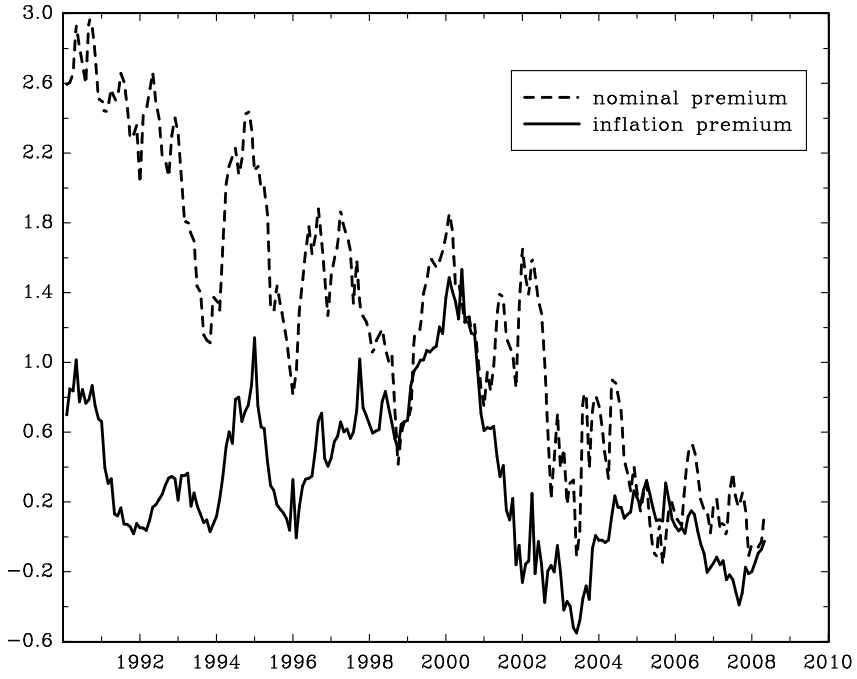



Figure 5b: Estimated 10-year euro area premia

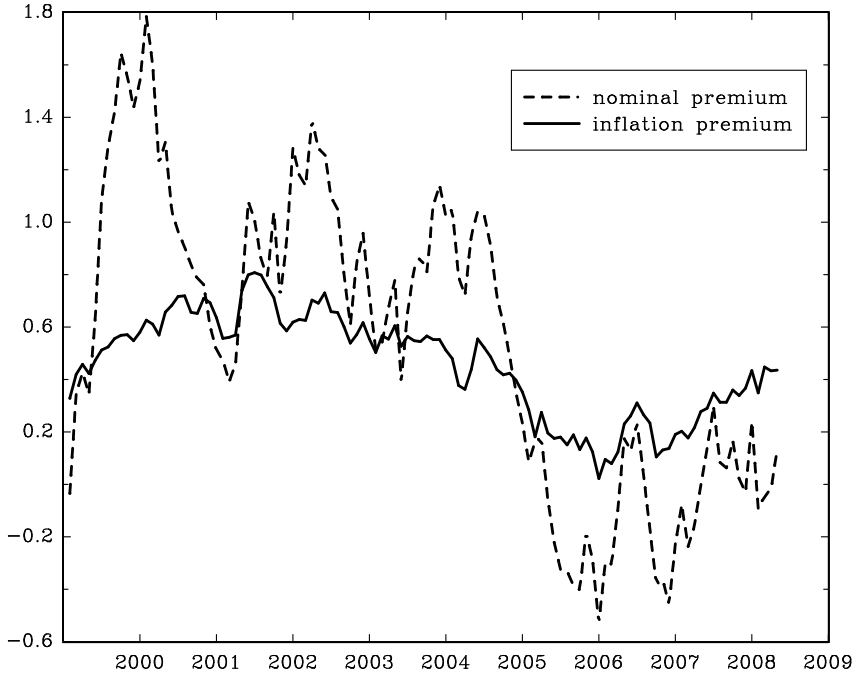


Figure 6a: US 10-year break-even inflation rates and survey inflation forecasts

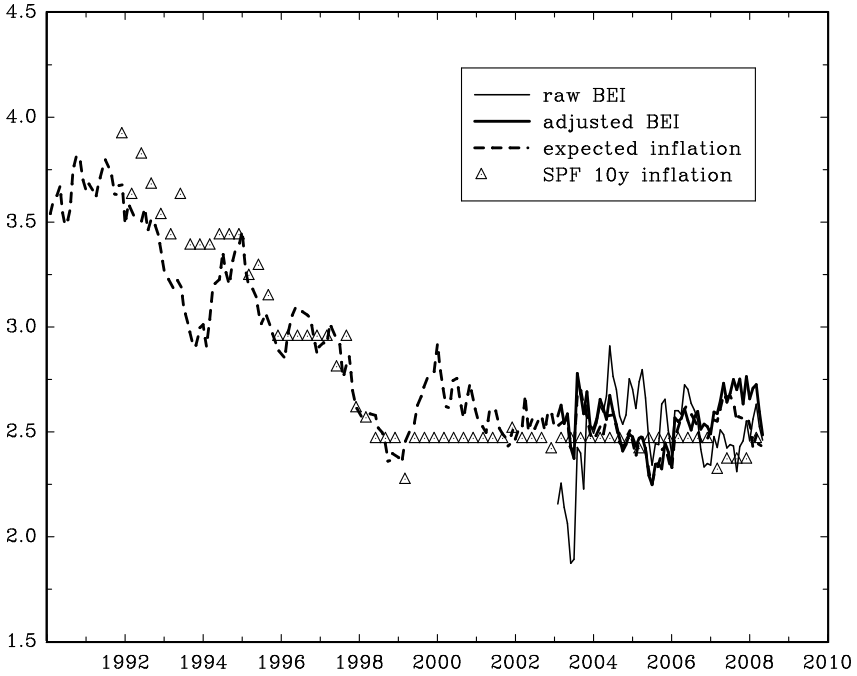
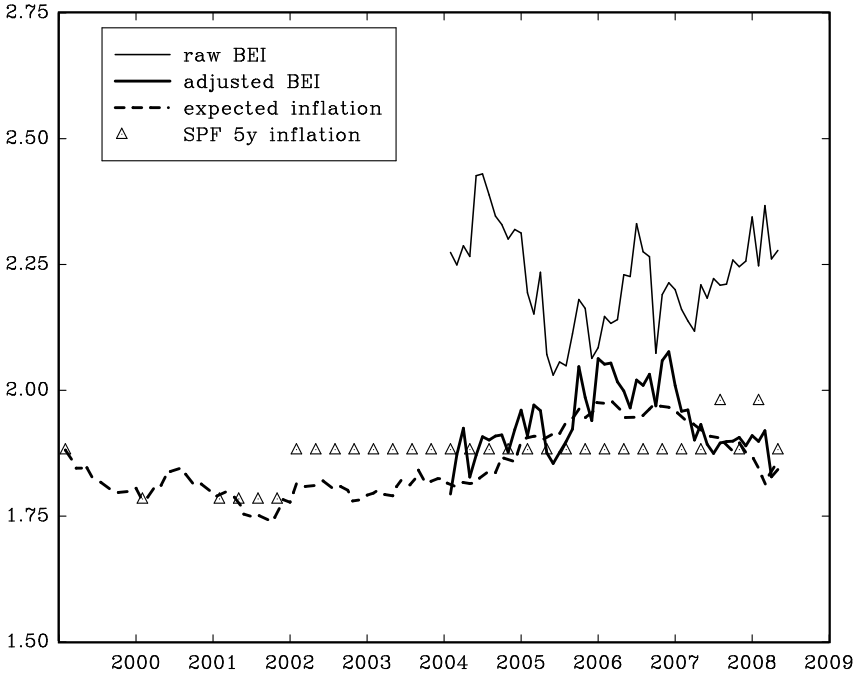



Figure 6b: Euro area 10-year break-even inflation rates and survey inflation forecasts




The solid thin line is the unadjusted model-implied 10-year break-even rate; the solid thick line is the break-even rate adjusted for the inflation risk premium; the dashed line is the model-implied average expected inflation over the next 10 years. The triangles are SPF survey expectations of inflation during the next 10 years (US) and five years ahead (euro area). All values are expressed in percent per year.

Figure 7a: US 10-year inflation risk premium and output gap

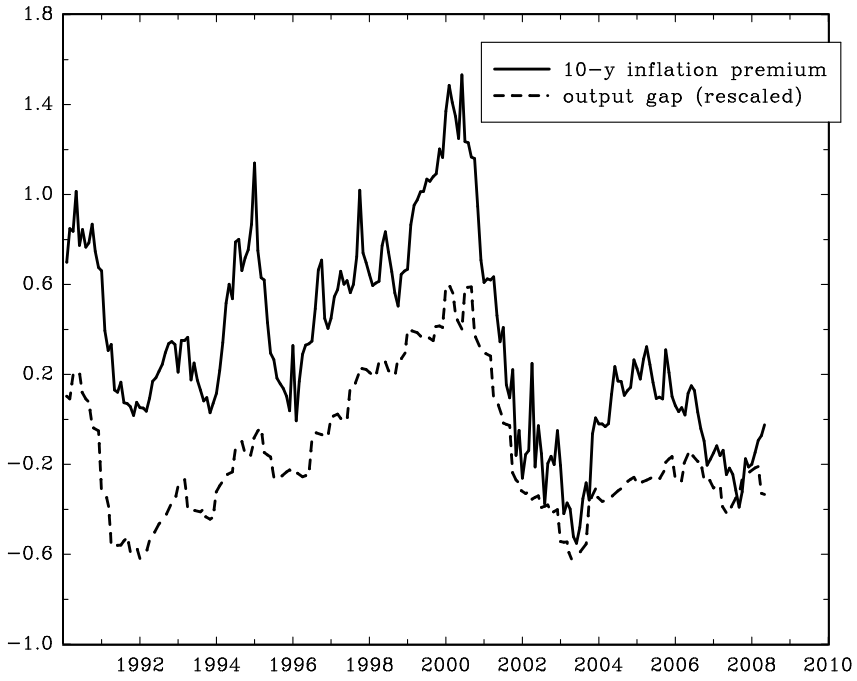



Figure 7b: US 10-year inflation risk premium and yoy inflation

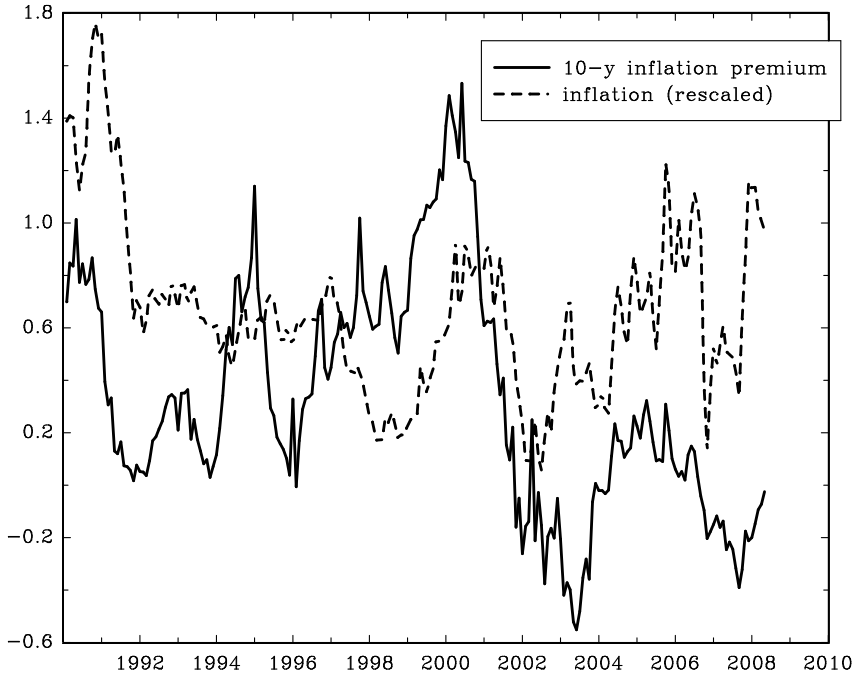


Figure 8a: Euro area 10-year inflation risk premium and output gap

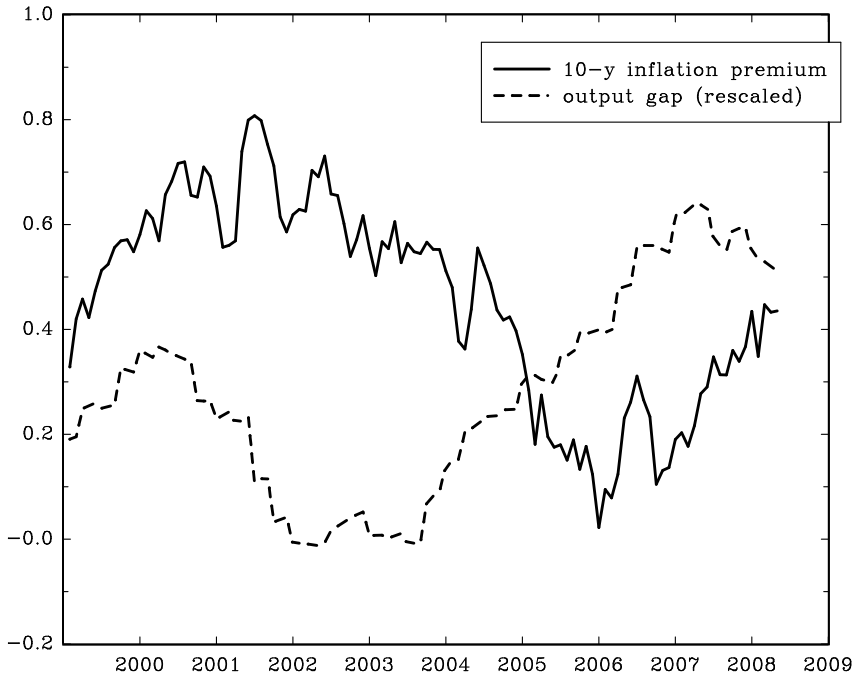



Figure 8b: Euro area 10-year inflation risk premium and yoy inflation

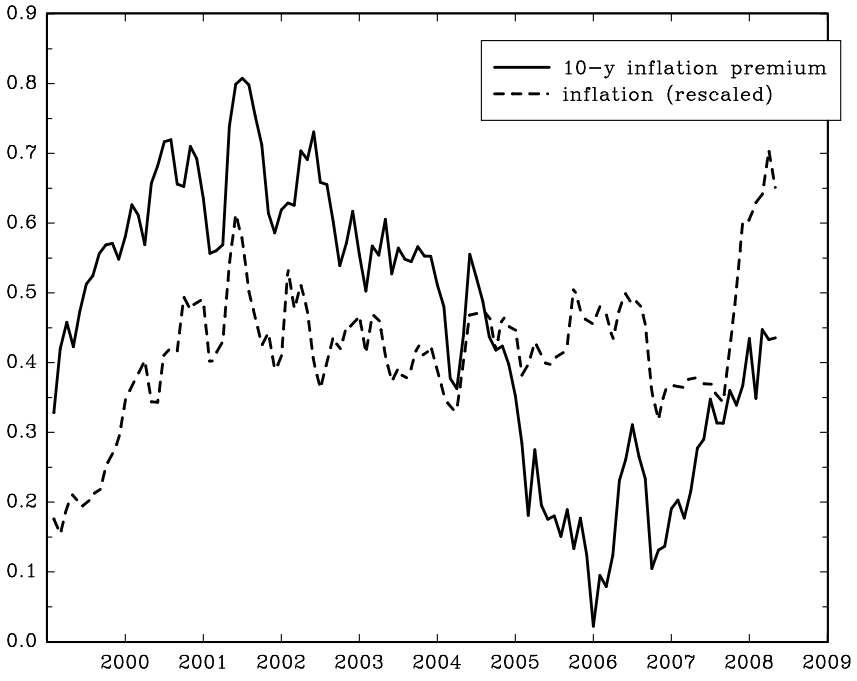


Figure 9: Responses to an output gap shock: US

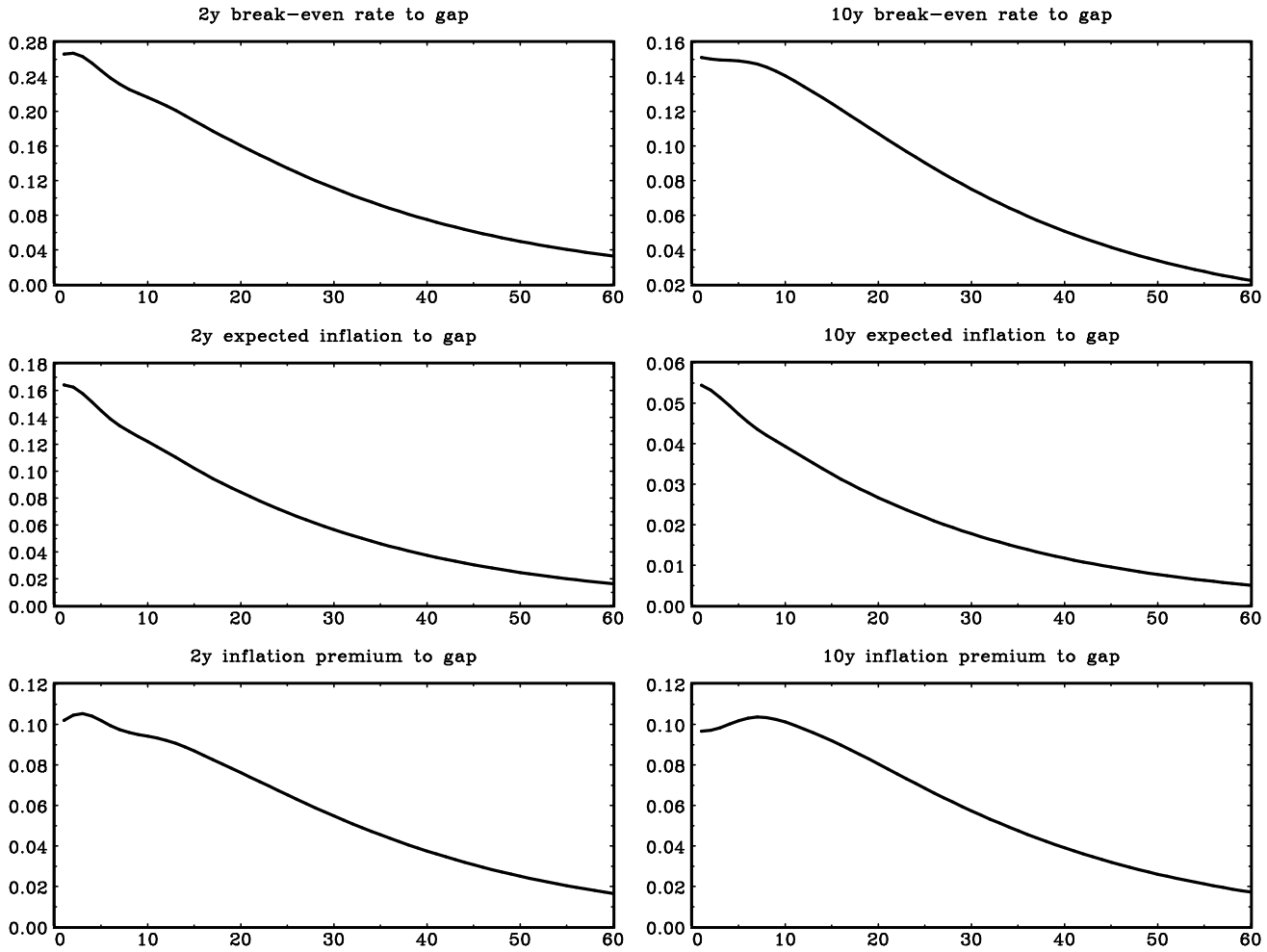


Figure 10: Responses to an inflation shock: US

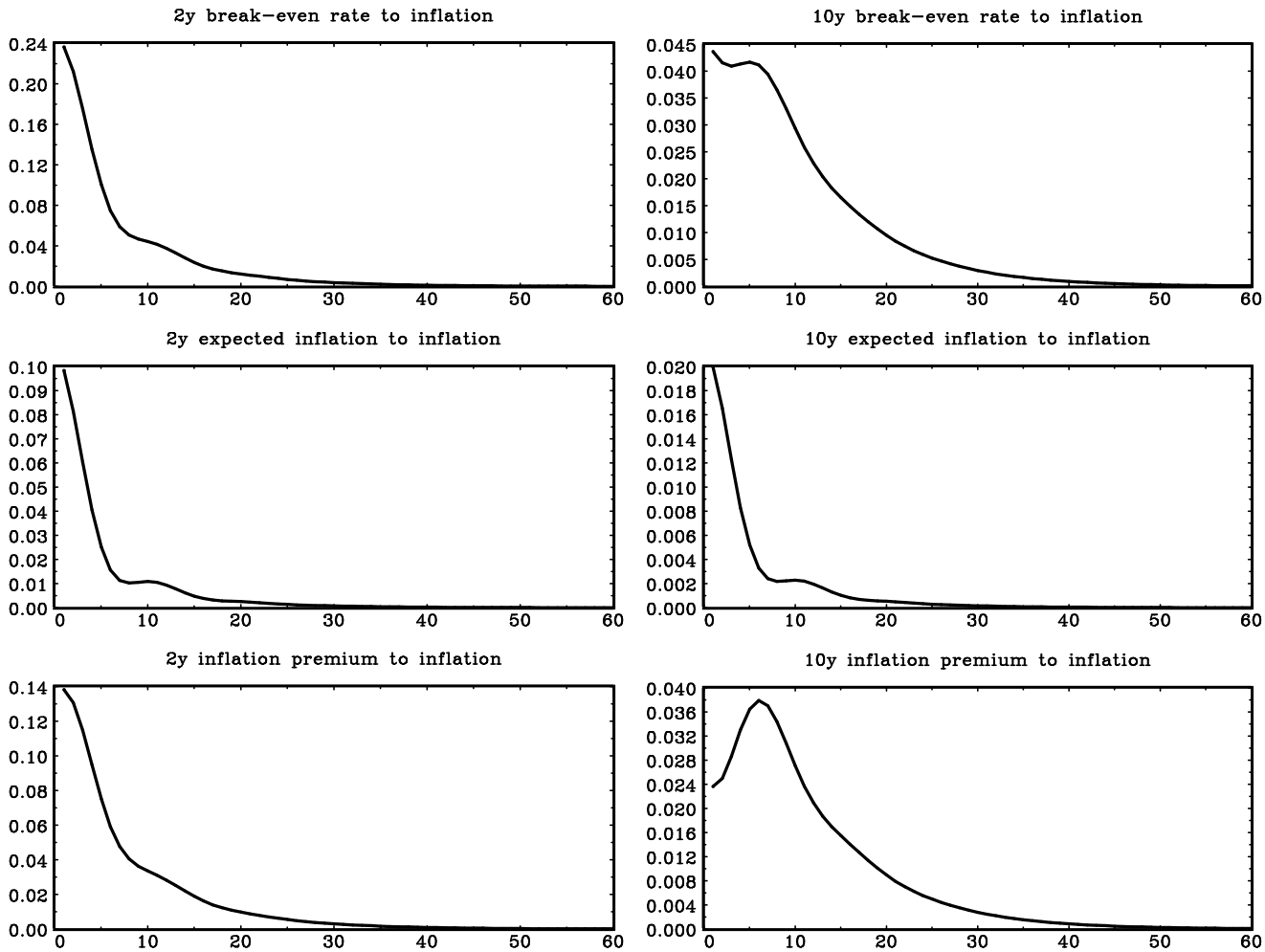


Figure 11: Responses to an output gap shock: euro area

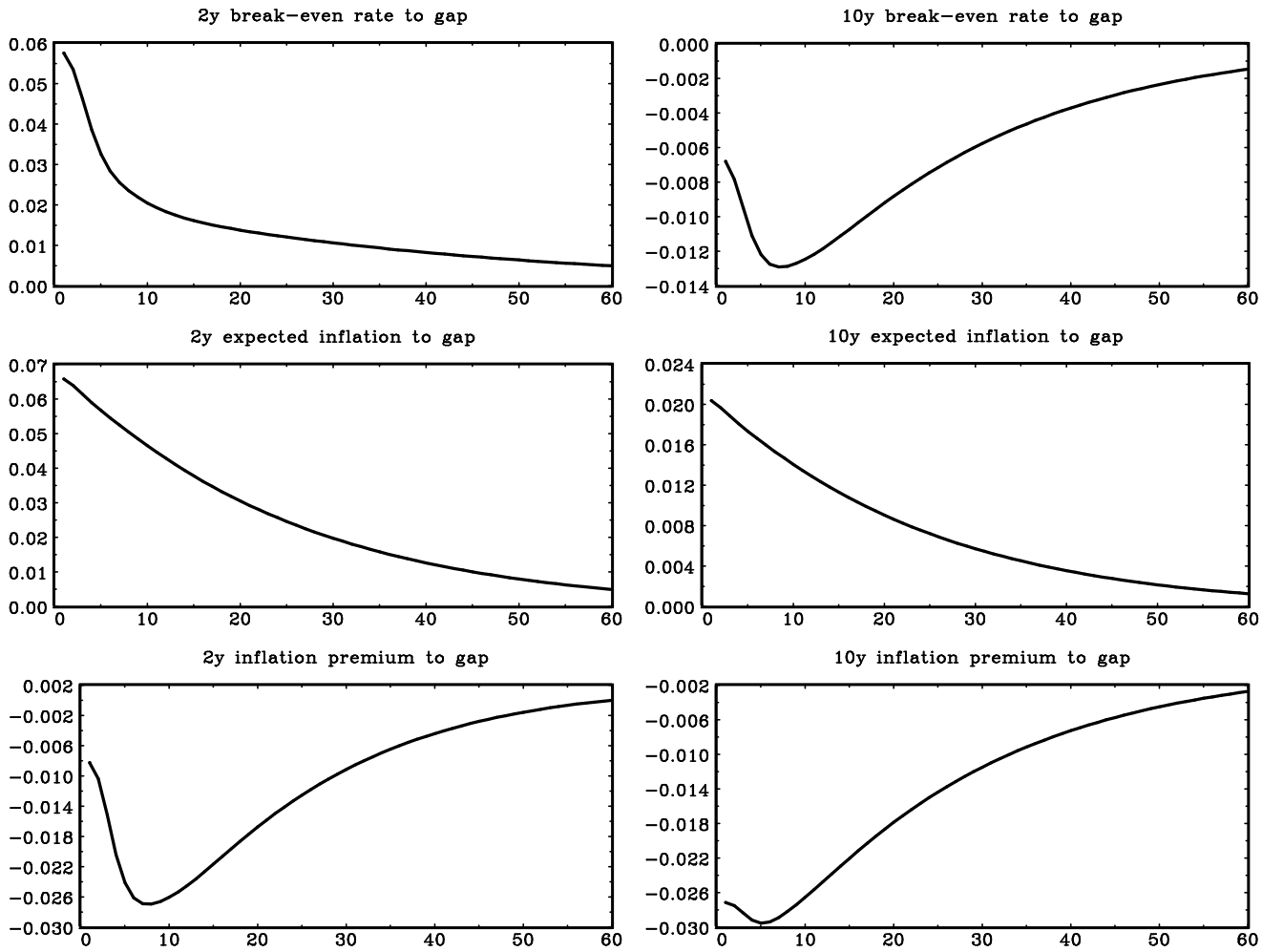


Figure 12: Responses to an inflation shock: euro area